L3 (IUM, Spring) Feb 23, 2021

Convex Polytopes

<u>Def</u> (elementary move)

$$P = \bigsqcup \Delta_i \xleftrightarrow{\text{elem. move}} \bigsqcup \Delta_i'$$

"all $\Delta_i = \Delta_i'$ except $2 \leftrightarrow 1$"

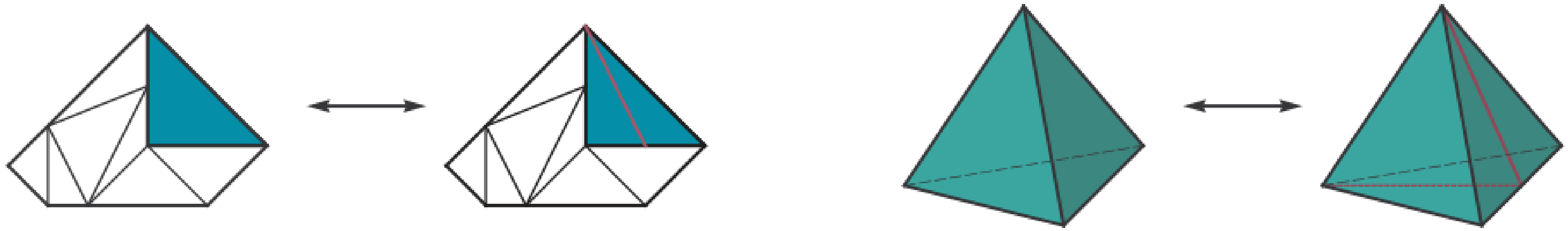


FIGURE 17.1. Examples of elementary moves on dissections.

Th [Ludwig - Reitzner, c.2006] (P - convex)

$\forall P \subset \mathbb{R}^d$ Every 2 simplicial dissections are connected by a finite seq of elem. moves.

$d=2$

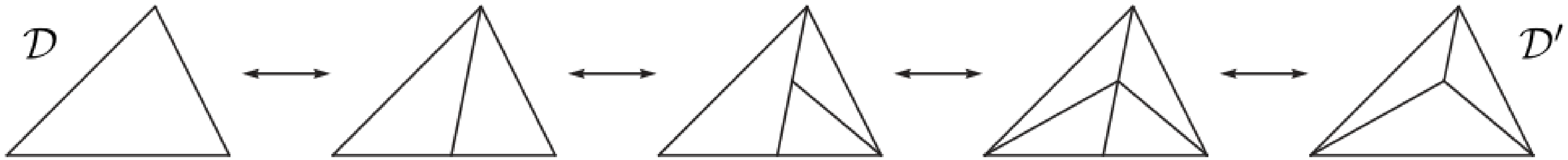
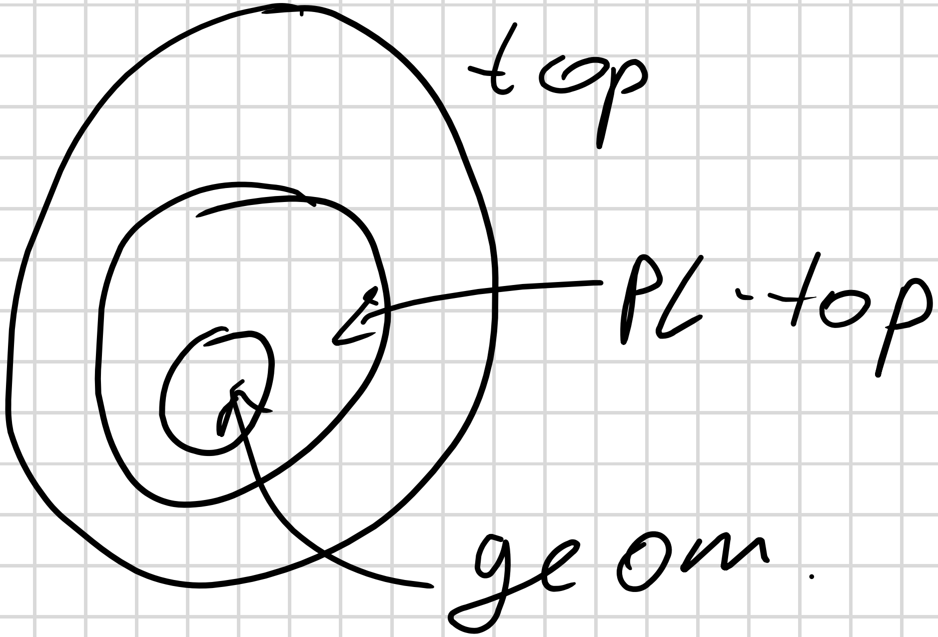
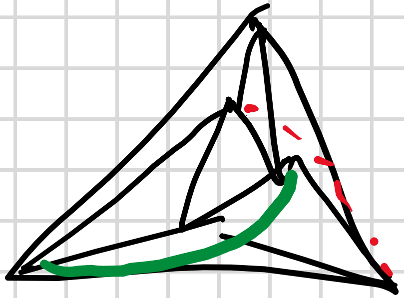


FIGURE 17.2. A sequence of elementary moves on dissections: $\mathcal{D} \leftrightarrow \mathcal{D}'$.

<u>Rem</u> geom. version of topol. thms





Def $\varphi : \{\triangle \subset \mathbb{R}^d\} \longrightarrow \mathbb{R}$

φ – *valuation* $\qquad \varphi(\triangle_1)+\varphi(\triangle_2) = \varphi(\triangle)$

φ – *symmetric* $\qquad \triangle \longleftrightarrow \triangle_1 \sqcup \triangle_2$

elem. move.

if $\varphi(g\triangle) = \varphi(\triangle)$

$g \in$ *rigid motion*

$\left\{ \begin{array}{l} \text{parallel transl,} \\ \text{rotation, reflection} \end{array} \right.$

Prop $\varphi \leftarrow$ sym. valuation

$\Rightarrow$ φ extends to all convex $P \subset \mathbb{R}^d$

$\square$ $P = \bigsqcup \triangle_i = \bigsqcup \triangle'_i$

$$\left.\begin{aligned} \varphi(P) &:= \sum \varphi(\triangle_i) \\ &= \sum \varphi(\triangle'_i) \end{aligned}\right\} \text{ by } \underline{\text{L-R}} \text{ } \underline{\text{thm}}$$

<u>Ex</u> $\varphi(P) := \mathrm{vol}(P)$ | $\exists\, \varphi \leftarrow$ sym val s.t. $\varphi(\square) \neq \varphi(\triangle)$ $\Rightarrow$ Dehn Th

Def $f : \mathbb{R} \to \mathbb{R}$
1) f – additive
2) $f(\pi) = 0$

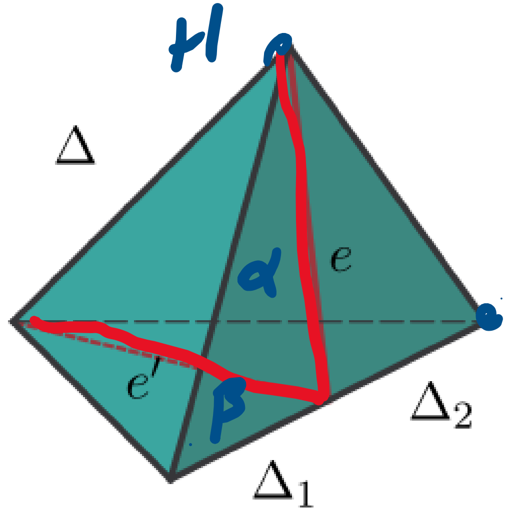
$\left.\right\}$ Kagan function.

Def $\forall f \in$ Kagan function $\quad$ $d=3$

$$\varphi = \varphi_f(\Delta) := \sum_{e \in \Delta} \underline{\ell_e \cdot f(\gamma_e)}$$

L $\varphi \in$ sym. valuation

D sym ✓ $\quad$ valuation $\longrightarrow$



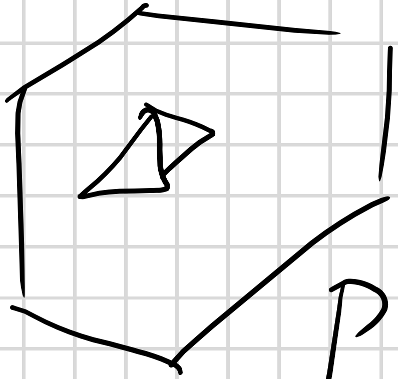
$$\Delta \longleftrightarrow \Delta_1 \sqcup \Delta_2$$

$$-\varphi(\Delta) + \varphi(\Delta_1) + \varphi(\Delta_2)$$

$$= \boxed{\ell_e f(\alpha) + \ell_e f(\pi-\alpha)} + \boxed{\ell_{e'} f(\beta) + \ell_{e'} f(\pi-\beta)}$$

$$= \ell_e f(\pi) + \ell_{e'} f(\pi) = 0 \qquad \square$$

<u>Cor</u> $\varphi_f(P) := \sum_{e \in P} \ell_e f(\theta_e)$ $\leftarrow$ well defined.



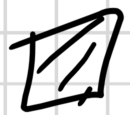
P

$$P = \bigsqcup_{i=1}^{N} \Delta_i$$

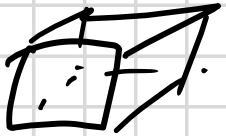
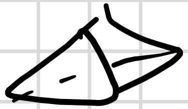
$$\varphi(P) = \sum_{i=1}^{N} \varphi(\Delta_i) \leftarrow \text{well def.}$$

$$= \sum_{i=1}^{N} \sum_{e \in \Delta_i} \ell_e \, f(\gamma_e) =$$

$$\boxed{\varphi_f(P) = \sum_{e \in E(P)} \ell_e \, f(\gamma_e)} \qquad \underline{\forall f} \leftarrow \text{kagan function}$$

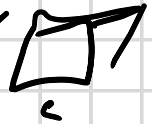


Dehn invariant.

Cor  $\not\sim$  $\alpha := \arccos \frac{1}{3}$

Exc $\exists$ kagan function

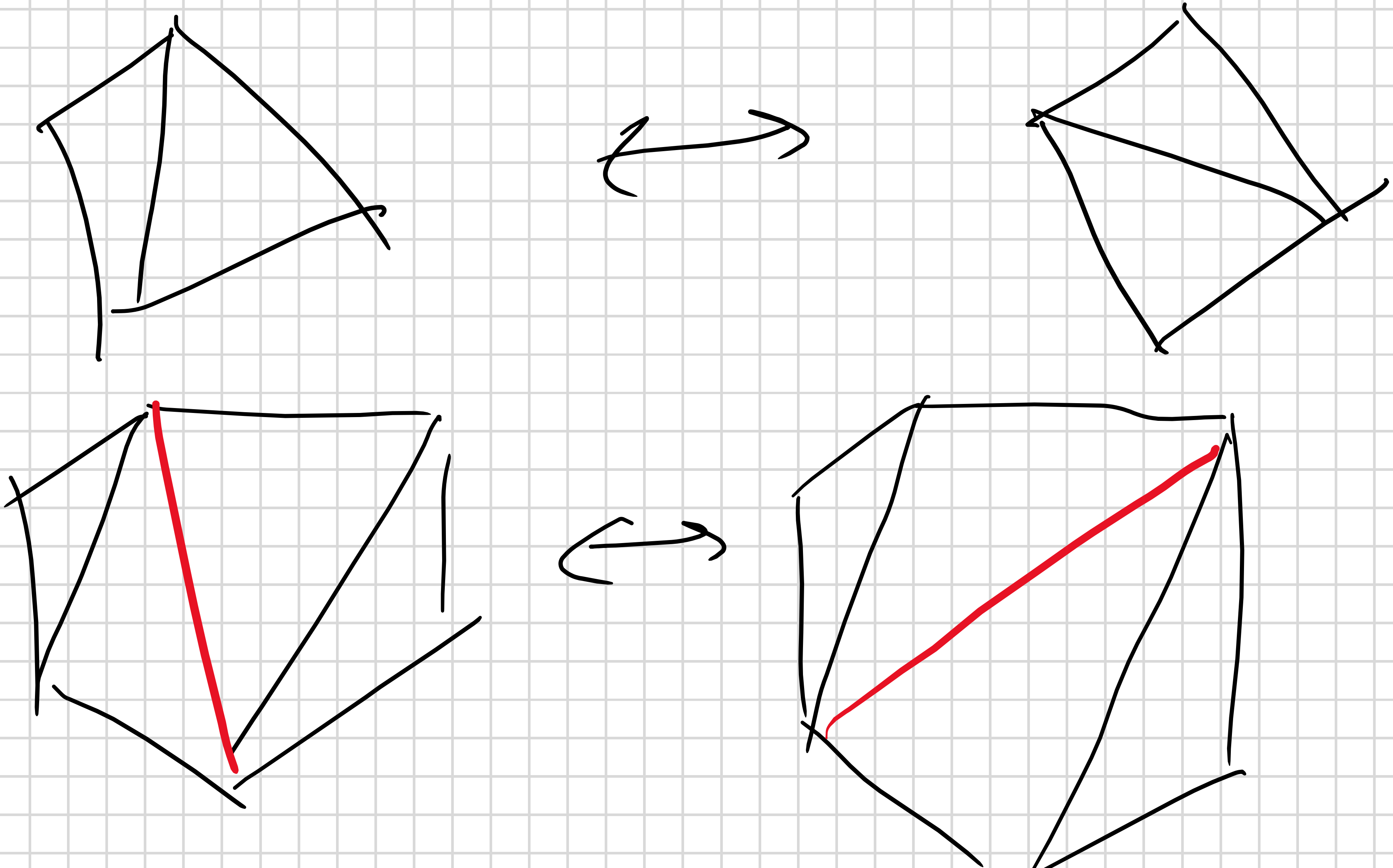
$$\left.\begin{aligned} f(\alpha s) &= s \\ f(\pi s) &= 0 \end{aligned}\right\} \quad \forall\ s \in \mathbb{Q} \qquad \alpha/\pi \notin \mathbb{Q}$$

D $\varphi($  $) = 12 \cdot c \cdot \varphi\left(\frac{\pi}{2}\right) = 0$

$\varphi($  $) = 6 \cdot 1 \cdot \varphi(\alpha) = 6 \neq 0$ $\square$

Now: proof of C-R

Def 2-move



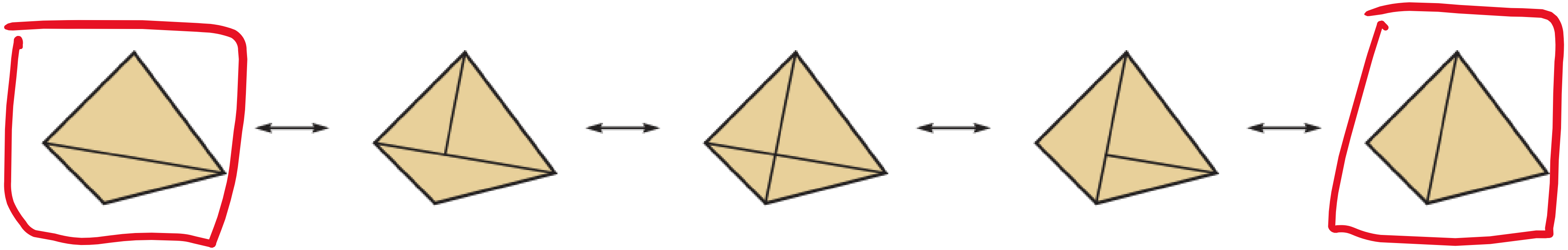
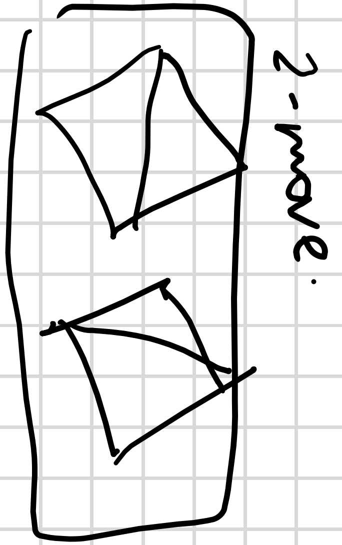
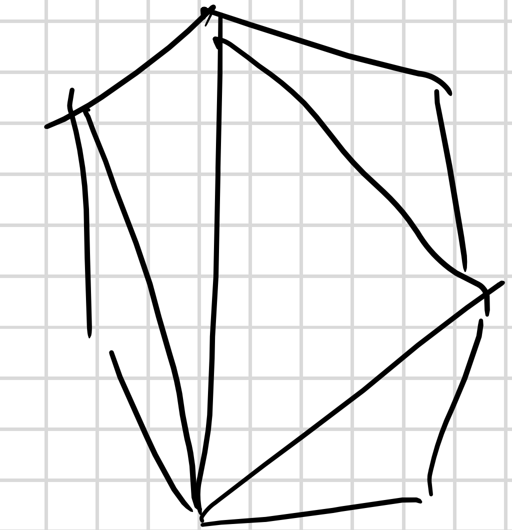
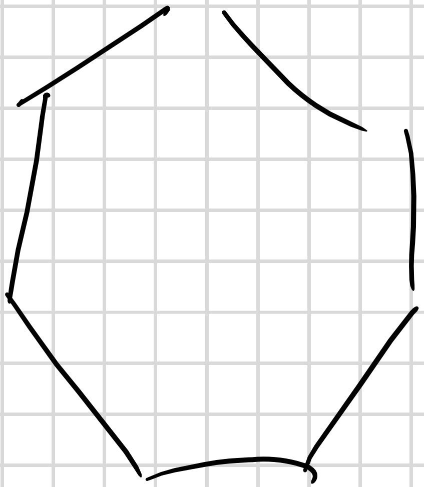


FIGURE 17.4. Making a 2-move using four elementary moves.



L1 Every 2 triangulations of convex polygon are conn w/ 2-moves

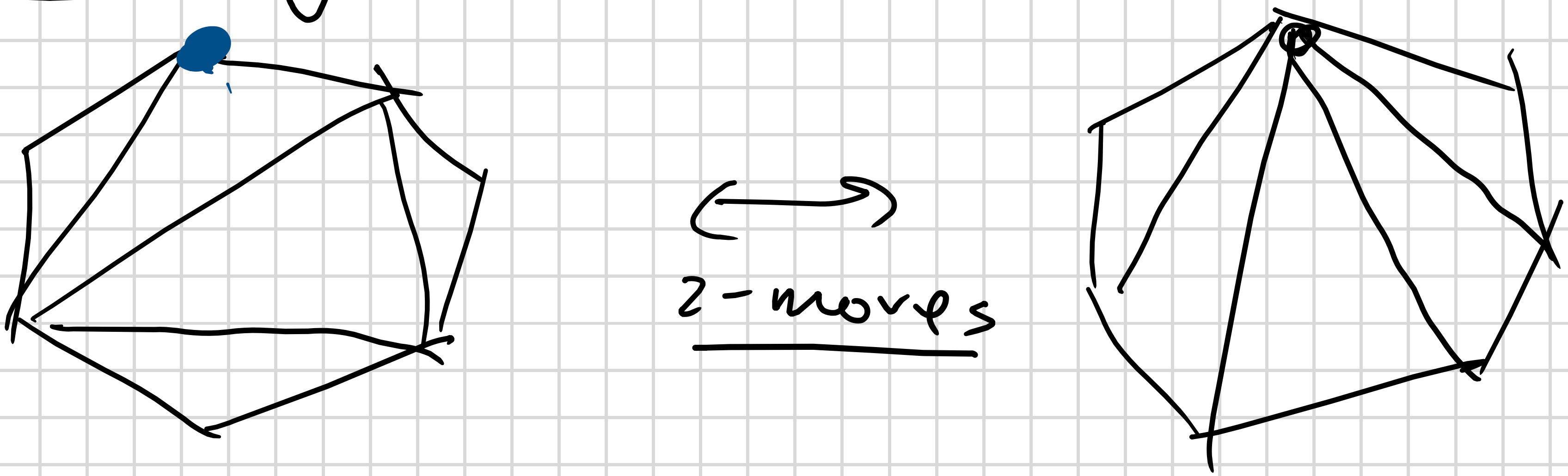




$\Rightarrow$ L2 same w/ elem. moves.

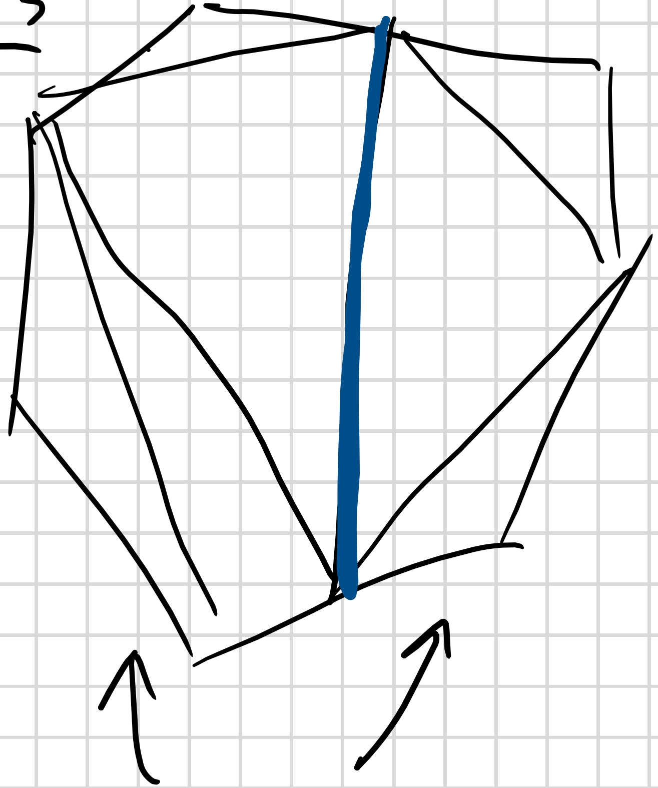
$L1 \Rightarrow L2$ ← means every 2-move is a sequence of 4 elem. moves.

Prog of L1



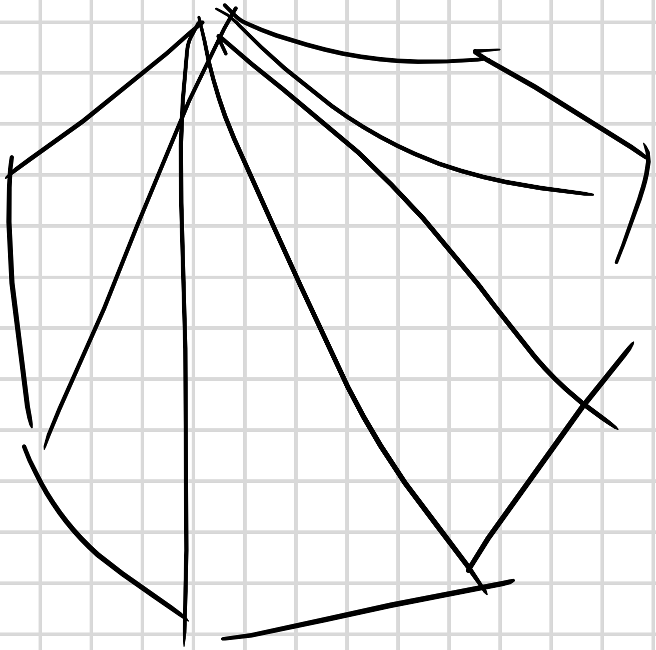
peg star triang.

L3



$\longleftrightarrow$

elem. move



star triangulations

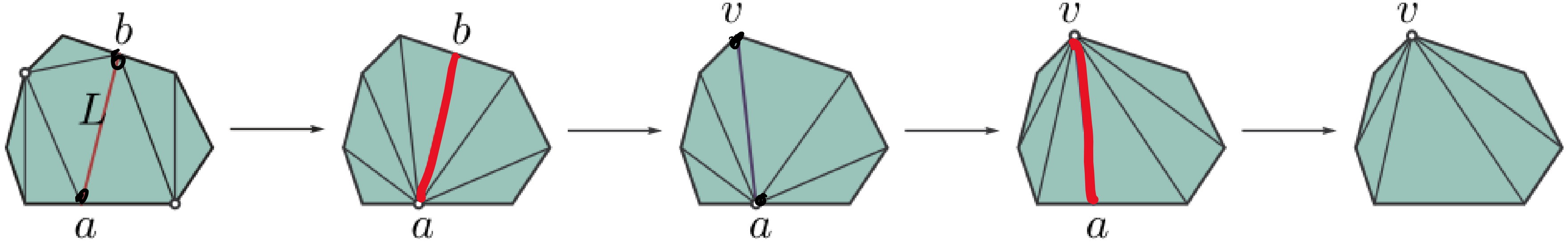
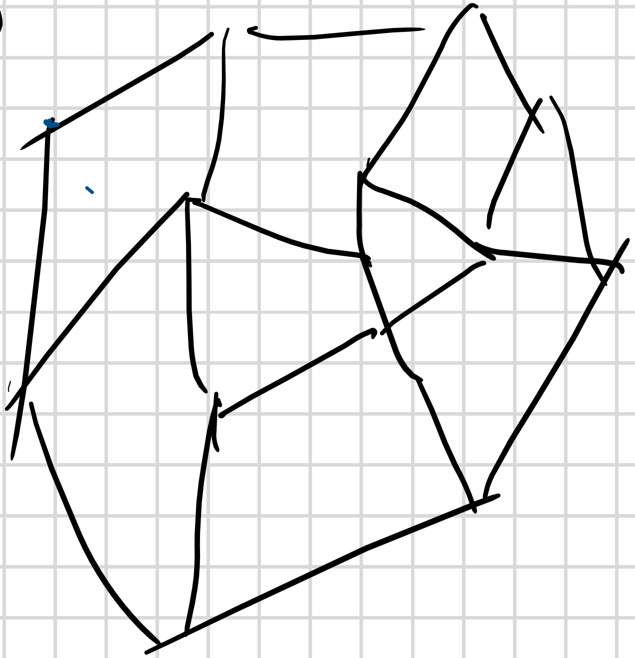


FIGURE 17.5. From two star triangulations to one.

Proof of L-R thm for $d=2$
(by induction on # polygons in polyhedral subdivision)

D



← polygonal subdivision Q

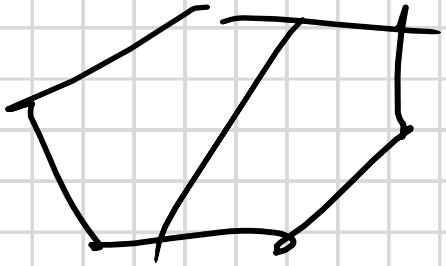
{star triang of Q}

claim all polygonal star triang are conn w/ elem. moves

{[induction on # polygons] ← n }

BASE n = 1 ✓

n = 2



✓

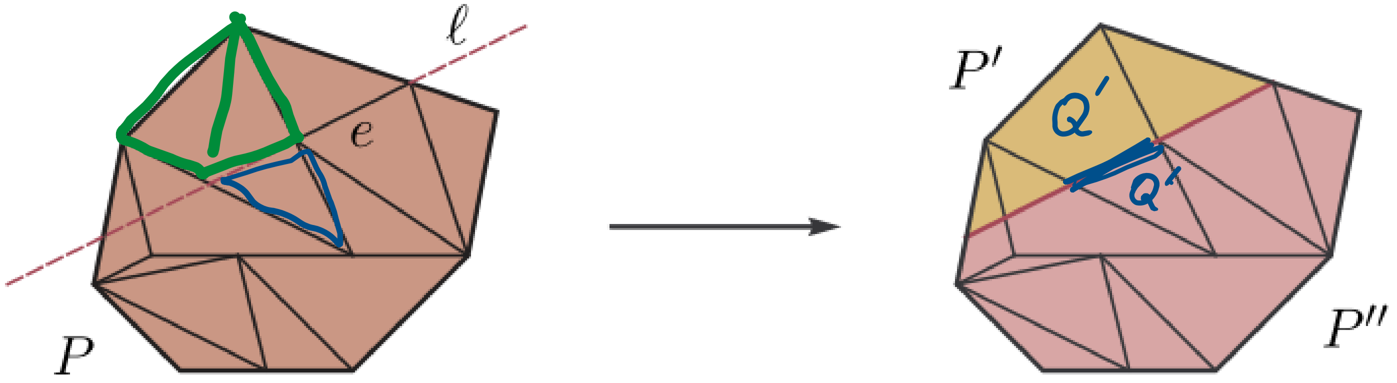


FIGURE 17.6. Cutting a decomposition by a line in the induction step.

STEP $n > 2$

OBS: # polygons in P > # polygons in P' and in P''

$Q' \cup Q'' \leftarrow Q = Q' \cup Q''$

$\blacksquare$

$d=3$

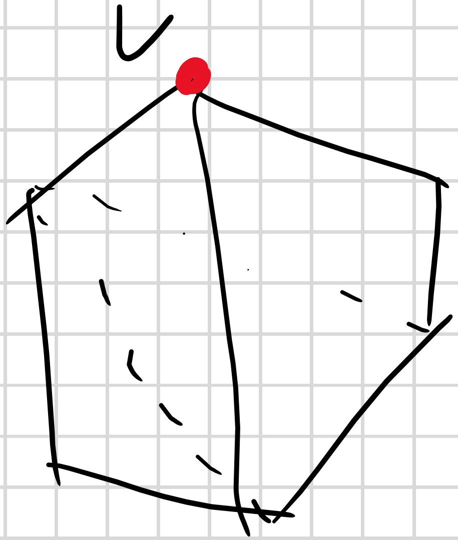
L.1 P - convex polytope

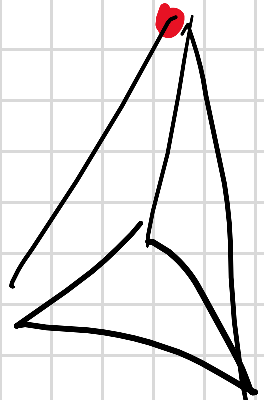
$\exists$ star triang $\leftarrow$ all Δ have a common vertex

▷ $v \in V(P)$

cone over all faces.

& triangulate all faces $\square$



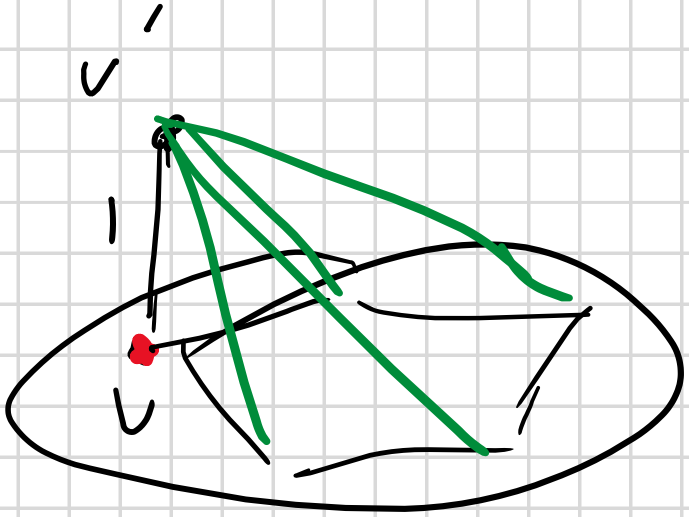


Def / Construction

$P \subset \mathbb{R}^d$, $v \in V(P)$

$P \subset \mathbb{R}^{d+1}$

$\mathbb{R}^{d+1}$

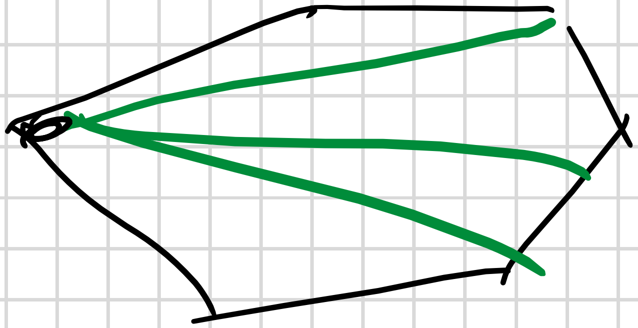


$P \subset \mathbb{R}^d$

$(vv') \perp \langle P \rangle$

$\mathrm{conv}\{P, v'\}$

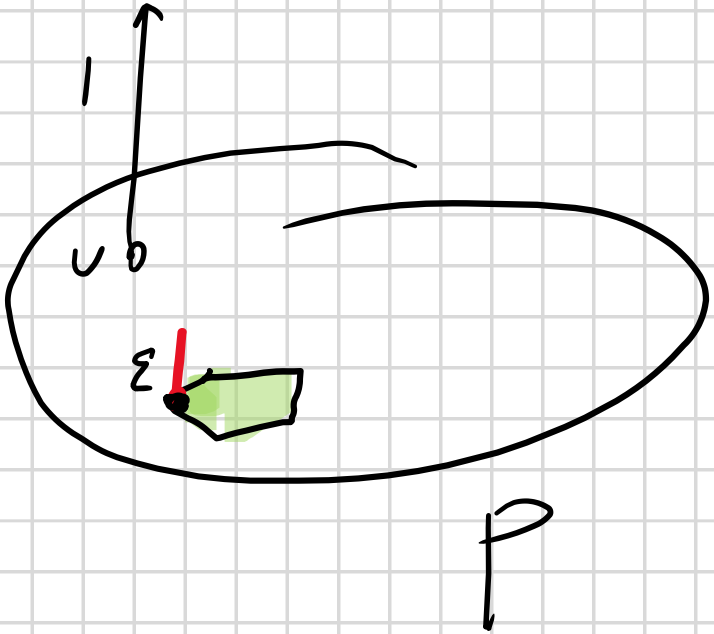
project upper convex hull onto P



star triang of P

L2 ⟸ L1   all star triang of $P \subset \mathbb{R}^3$
are connected by elem. moves

regular triang $h: V \to \mathbb{R}_+$ ← height function
/coherent/



h – generic

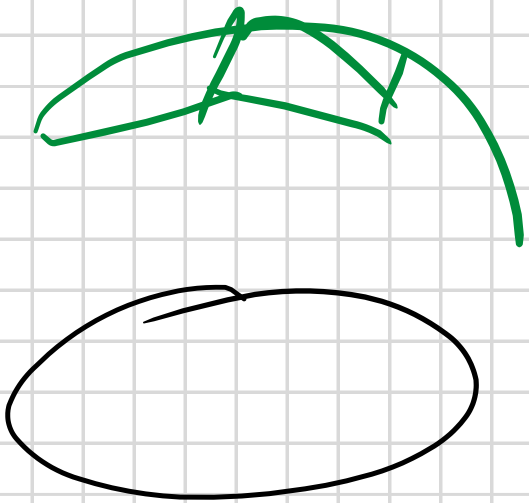
conve. hull of $\{(v, h(v)) \mid v \in V(P)\}$ with $(v, h(v)) \in \mathbb{R}^3$, $h(v) \in \mathbb{R}$

$\mathbb{R}^3 \downarrow$

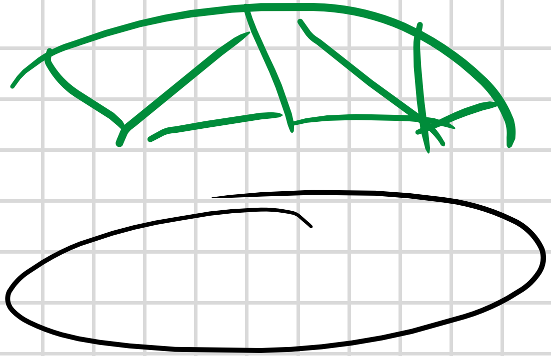
triang of P

$h: V \to \mathbb{R}_+$

$h': V \to \mathbb{R}_+$



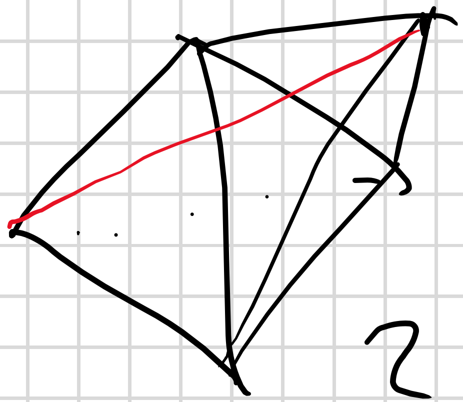
p

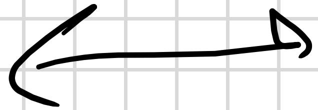


p

$h \longrightarrow h'$

$h, h' \in \mathcal{M} \in$ generic h. functions



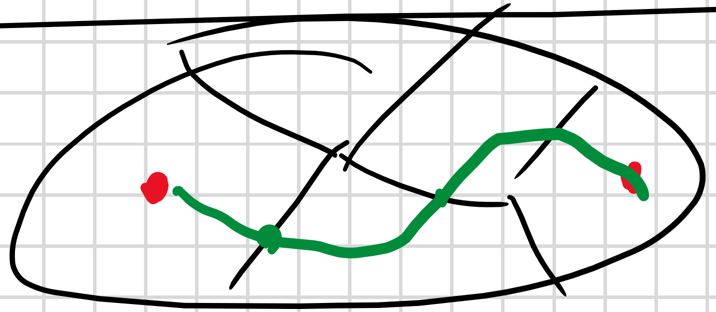


2   3

<u>L/Exc</u>

<u>2-3 transf.</u> is a seq of elem. moves

M



$\blacksquare$