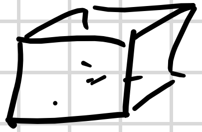
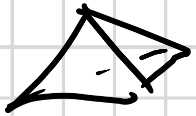
Polytope Algebra

Last time

Th (Dehn)  $\not\sim$ 

Th (Sydler 1944)  $\not\sim$  + ... +  ← all regular

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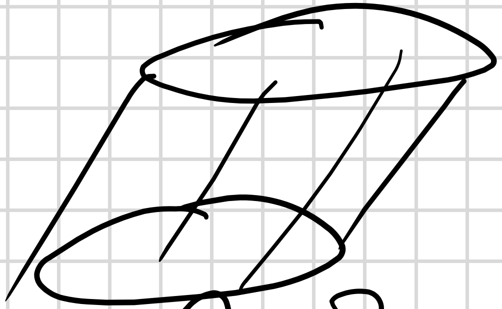
Def $P \subset \mathbb{R}^d$ ← rectifiable

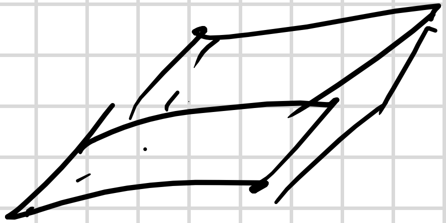
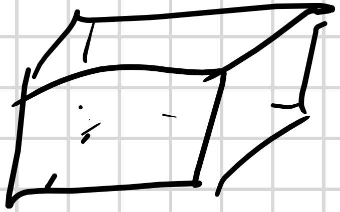
if $P \sim$  ← cube of vol(P)

Obs all prysms in $\mathbb{R}^3$ are rectif.

 $\leftarrow Q + v\cdot\underbrace{[0,1]}_{I}$ $\leftarrow$ Minkowski sum

2

Q 2

 $\sim$  $\leftarrow$ cube.

$\mathbb{R}^3$

L $P \subset \mathbb{R}^3$ convex pol.

$\alpha_1, \alpha_2, \dots, \alpha_k > 0$ $\quad \alpha_1 + \dots + \alpha_k = 1$

$\Rightarrow P \sim \alpha_1 P \oplus \alpha_2 P \oplus \dots \oplus \alpha_k P \oplus R$ where $R \in \mathcal{R}$

here $c \cdot P$ = expansion of P w/ const c /dilation/

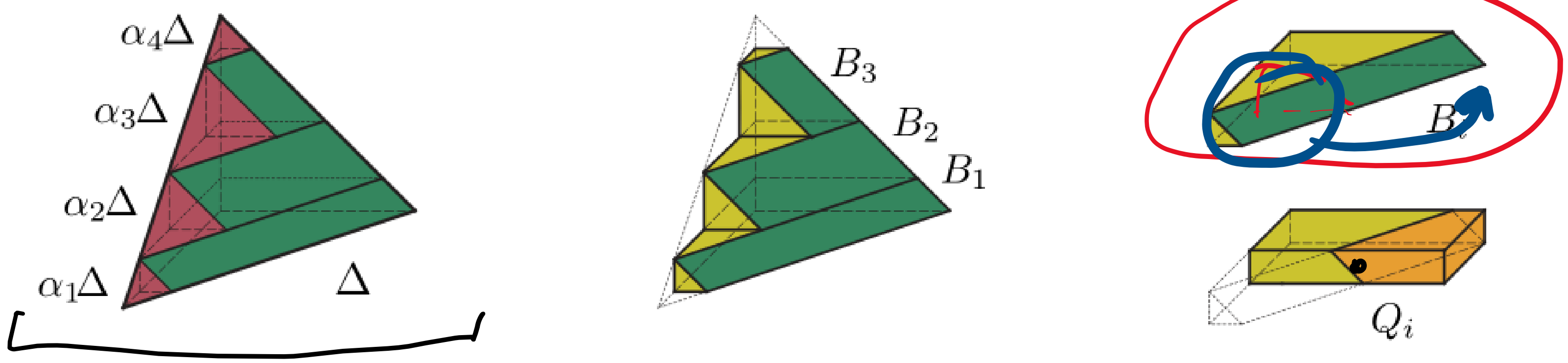


FIGURE 16.1. Layers $B_i \cup \alpha_i\Delta$ and scissor congruence $B_i \sim Q_i$.

Proof

$P = \bigsqcup \Delta_i$

$\Rightarrow$ suffices to prove for Δ

box

$$\Delta = \alpha_1\Delta \oplus \dots \oplus \alpha_k\Delta \oplus R$$

"$\{x+y+z \leq 1, \ x, y, z \geq 0\}$

We use $R_1, R_2 \in \mathcal{R} \Rightarrow R_1 \oplus R_2 \in \mathcal{R}$

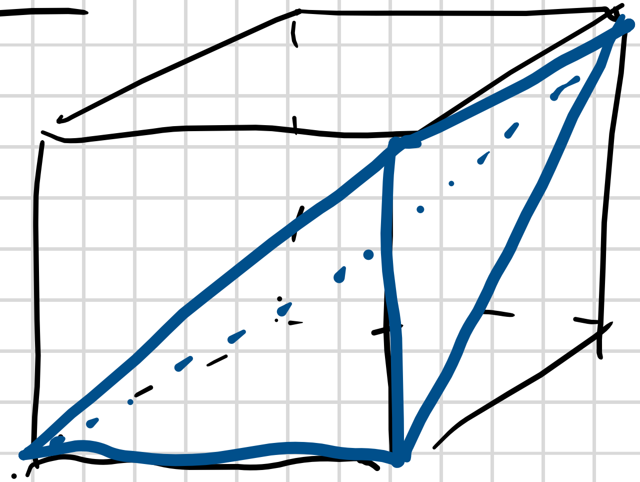
Th (tiling thm)

$P \subset \mathbb{R}^3$ s.t. $\boxed{P_1 \oplus \ldots \oplus P_m \in \mathcal{R}}$ ✓

where $P_i \leftarrow$ either P or $-P$

Then $\boxed{P \in \mathcal{R}}$

Ex



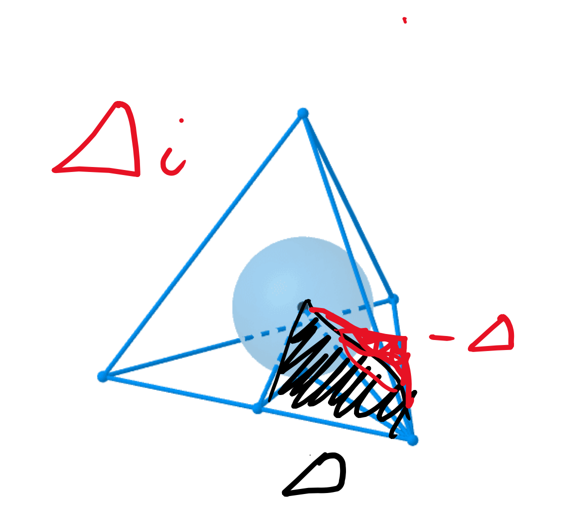
Δ_0

Cor $\Delta_0 \in \mathcal{R}$

6 copies of Δ_0 tile 

Prop $\forall P \sim -P$

(even if we use only rotations)

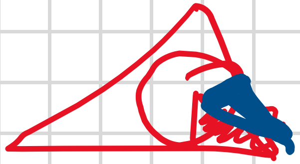


$$\Rightarrow \quad P = \bigsqcup \Delta_i$$

$$-P = \bigsqcup -\Delta_i$$

[Bricard]

$$\Delta_i \sim -\Delta_i$$



Proof $\quad m \times P := \underbrace{P \oplus P \oplus \dots \oplus P}_{m} \in \mathcal{R}$

$$\left\{\begin{array}{l} m \times P \sim R_1 \in \mathcal{R} \\ mP \sim (m \times P) \oplus R_2 \end{array}\right\} \quad mP \sim (m \times P) \oplus R_2 \sim \underbrace{R_1 \oplus R_2}_{} \in \mathcal{R}$$

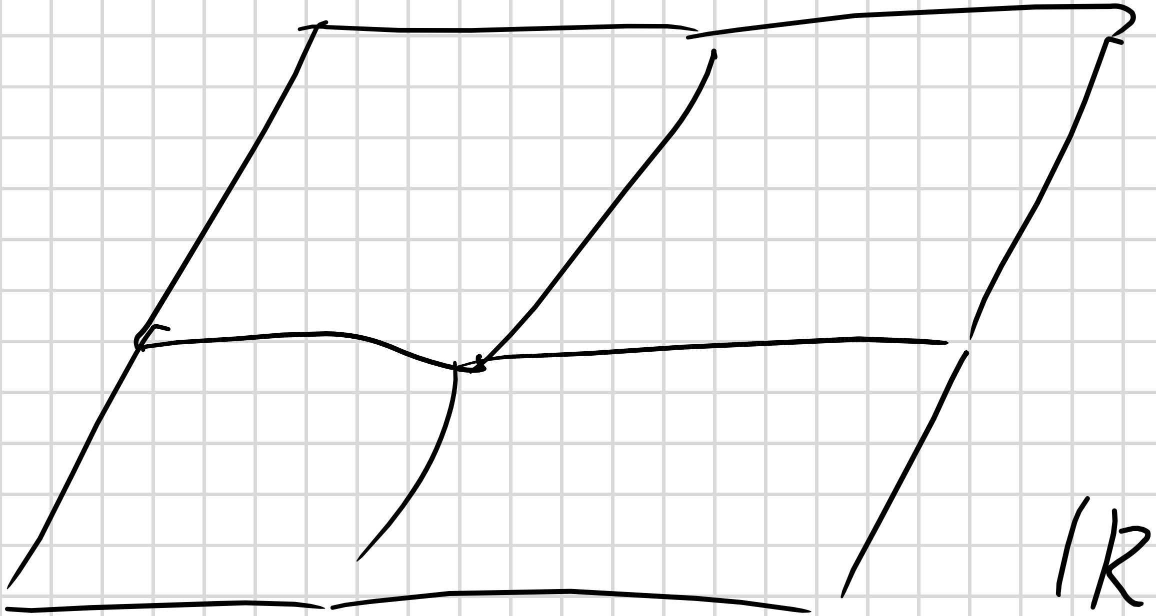
↑ Lemma

$\square$

Cor $P \subset \mathbb{R}^d$ tiles $\mathbb{R}^d$ periodically

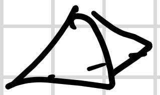
$\Rightarrow P \in \mathcal{R}$

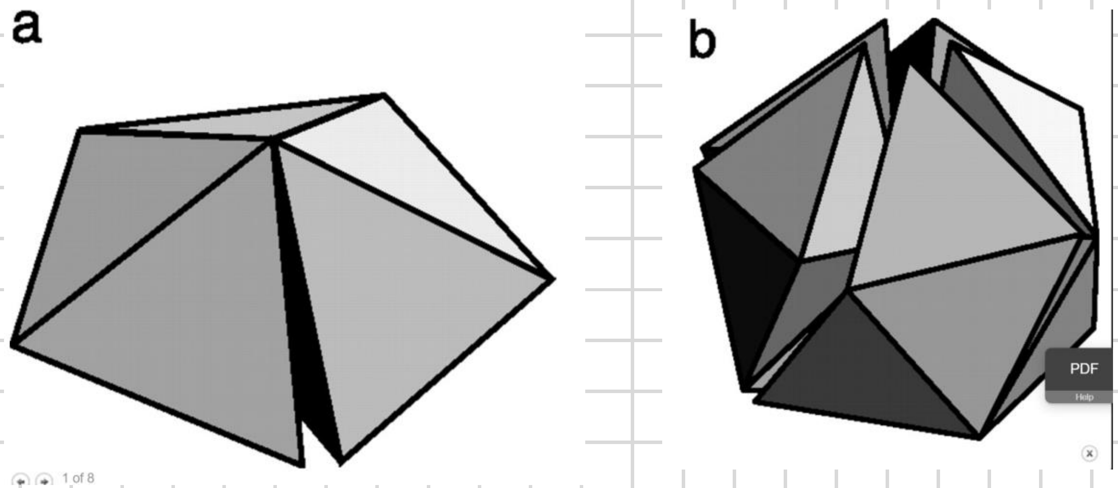
$D \cap P \sim \underline{R} \in \mathcal{R} \Rightarrow P \in \mathcal{R}$ $\boxtimes$



$\mathbb{R}^d$

$\mathbb{Z}^d$ acts on $\mathbb{R}^d$

Cor  regular does not tile $\mathbb{R}^3$

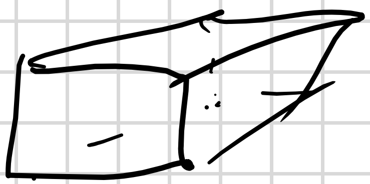


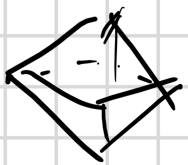
Aristotle : yes

One can tile $\mathbb{R}^3$

Cor 2 Q_4 = 4-dim cross polytope.

= conv hull of $\{\pm e_i\}$, $i = 1 \ldots 4$





Obs Q_4 tiles $\mathbb{R}^4$ periodically.

Exc

$\Rightarrow_{Th}$ $Q_4 \in \mathcal{R}$

Th $A, B, C, D \in \mathbb{R}^3 \leftarrow$ convex polytopes

s.t. $\left.\begin{array}{l} A \oplus B \sim C \oplus D \\ \textbf{and} \quad B \sim D \end{array}\right\} \Rightarrow A \sim C$

(complementarity thm)

D 1) $\operatorname{vol} A = \operatorname{vol} C$

2) $A' := \frac{1}{n} A, \; \ldots \quad D' = \frac{1}{n} D$

Lemma $\left\{\begin{array}{l} A \sim \underbrace{(n \times A')} \oplus R_1 \\ C \sim \underbrace{(n \times C')} \oplus R_2 \end{array}\right.$ $\quad R_1, R_2 \in R$

$n \to \infty$

$\operatorname{vol} R_1 = \operatorname{vol} R_2 = \left(1 - \frac{1}{n^2}\right) \operatorname{vol}(A) \to \operatorname{vol}(A)$

$\Rightarrow \quad \underline{R_1} \sim \underline{(n \times B')} \oplus \underline{S}$, some $\underline{S \subseteq \mathbb{R}^3}$ $\oplus$

$$A \sim (n \times A') \oplus R_1 \sim (n \times A') \oplus (n \times B') \oplus S$$

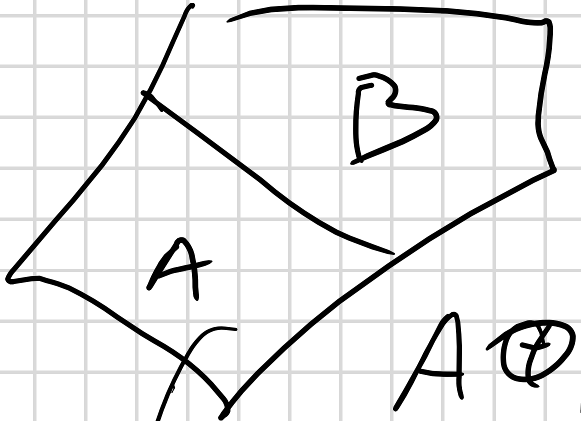
$$\sim n \times \underline{(A' \oplus B')} \oplus S \sim n \times (C' \oplus D') \oplus S$$

$$\sim (n \times C') \oplus (n \times D') \oplus S$$

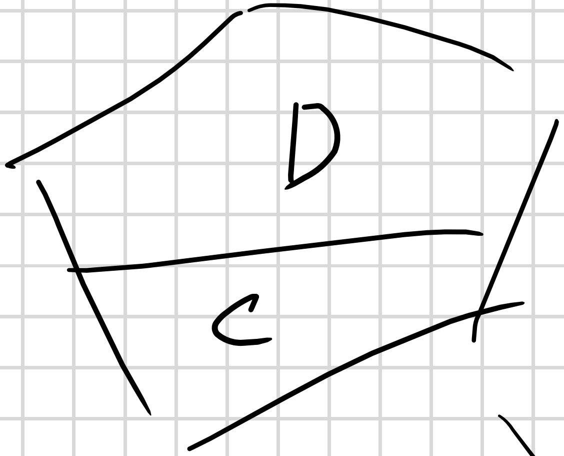
$$\sim (n \times C') \oplus \underline{(n \times B') \oplus S}$$

$$\sim (n \times C') \oplus \underline{R_1} \sim (n \times C') \oplus \underline{R_2} \sim C$$

$\boxtimes$



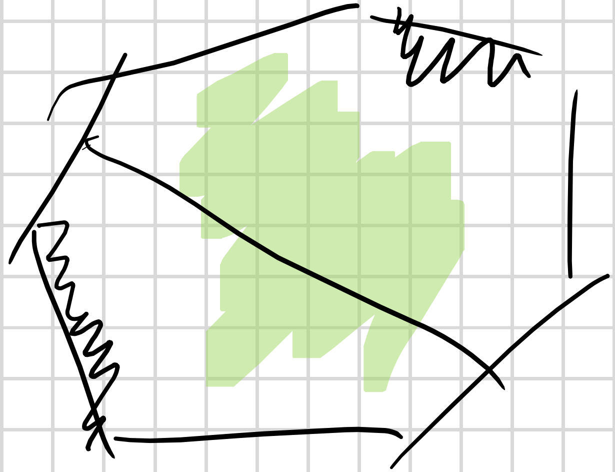
$A \oplus B$

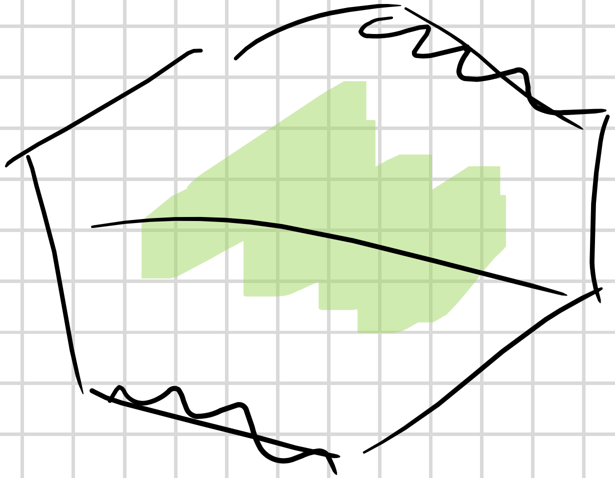


$C \oplus D$

$B \sim D$

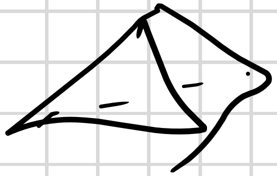
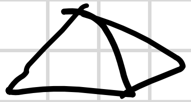
$A \oplus B \sim C \oplus D$





$\Longrightarrow$ <u>Th</u> (Sydler) $P \subset \mathbb{R}^3$ - conv. polg.

$$P \in \mathcal{R} \iff P \sim \underline{c\,P} \oplus R \quad , c<1$$
$$R \in \mathcal{R}$$

<u>Ex</u>  $\not\sim$ c  $\oplus\ \mathbb{R}$ $\quad , 0<c<1$

$\triangleright$ $cP \oplus R \sim P \underset{c}{\sim} cP \oplus (1-c)P \oplus R'$

$\Longrightarrow$ $R \sim (1-c)P \oplus R' \Longrightarrow$

<u>cc</u>

$\Longrightarrow P \in \mathcal{R}$ $\square$

Th (Sydler) $P \subset \mathbb{R}^3$

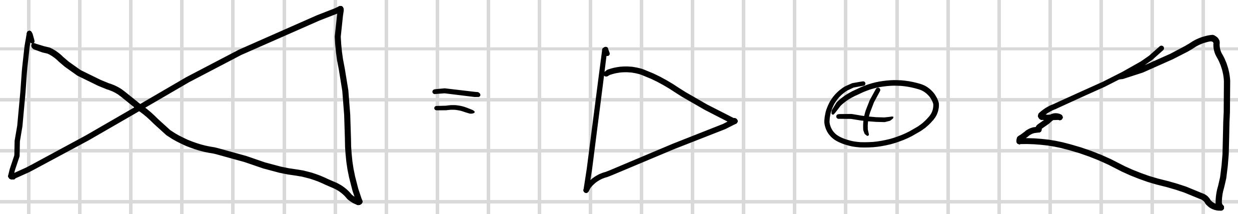
$$P \in \mathcal{R} \iff \exists \bar{c} \begin{cases} P \sim c_1 P \oplus c_2 P \oplus \dots \oplus c_k P \\ \underline{c_i > 0}, \quad k \geq 2 \end{cases}$$

$\bar{c} = (c_1 \dots c_k)$

D $\Leftarrow$

$$\begin{cases} P \sim c_1 P \oplus \dots \oplus c_k P, \quad c = c_1 + \dots + c_k \neq 1 \\ cP \sim c_1 P \oplus \dots \oplus c_k P \oplus R \qquad (\text{Lemma}) \end{cases}$$

$$\Rightarrow (c-1) P \sim R \Rightarrow P \in \mathcal{R} \quad \blacksquare$$



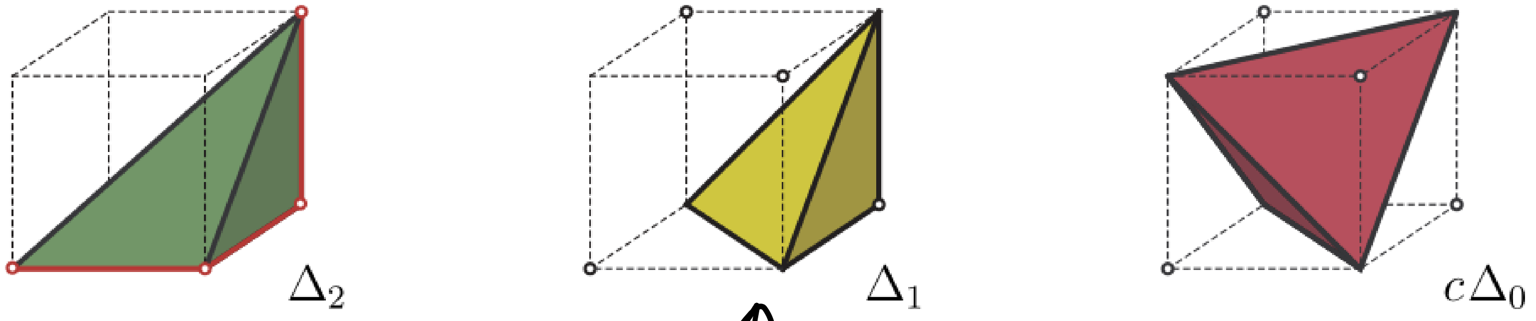


FIGURE 16.2. Hill tetrahedron, standard tetrahedron and the regular tetrahedron.

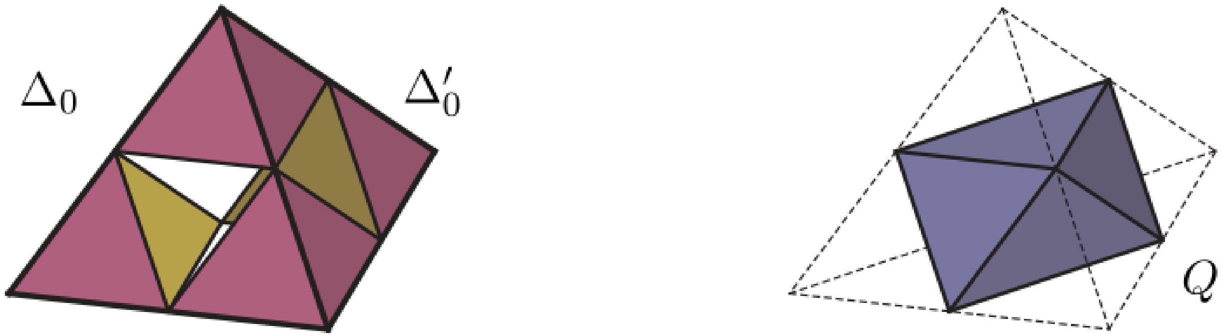
$\Delta_2 \in R \Leftarrow$ tiles cube.

$\Delta_0 \notin R \Leftarrow$ Dehn

$\Rightarrow \Delta_1 \notin R$

$\boxed{\Rightarrow \Delta_1 \not\sim c' \Delta_0}$

$(c\Delta_0 + 4\Delta_1) \in R \Rightarrow \Delta_1 \notin R$



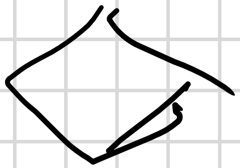
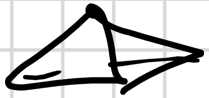


FIGURE 16.3. Octahedron Q and four tetrahedra Δ_0' tile tetrahedron Δ_0.

obs $\quad Q \not\sim c$  bc.

$Q + 4$  $= 2$ 

$\Rightarrow Q \notin R$

Exc $\quad Q_2, Q_4 \in R$

$Q_3 \notin R$

Th (Zyller)

$\forall P, Q \subset \mathbb{R}^3 \Rightarrow P \asymp Q$

Def ($\asymp$) $P \asymp Q$ if

$\exists\ P = \bigsqcup P_i\ ,\ Q = \bigsqcup Q_i$

s.t. $P_i \underset{\text{congruent}}{\sim} c_i Q_i$ some $c_i > 0$

Proof

Lemma $\left\{ \begin{array}{l} \triangle \sim 3\underline{\triangle'} \oplus R_1 \\ \triangle \sim 2\underline{\triangle''} \oplus R_2 \end{array} \right\}$ $\triangle' = \frac{1}{3}\triangle$, $\triangle'' = \frac{1}{2}\triangle$

$\Rightarrow \triangle' \in R_{\asymp} \Rightarrow \triangle \in R_{\asymp}$ ☑

①

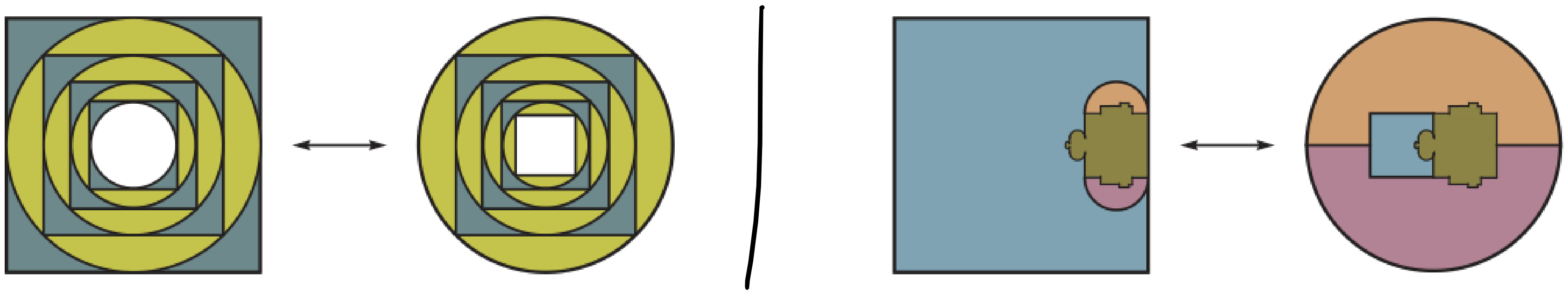
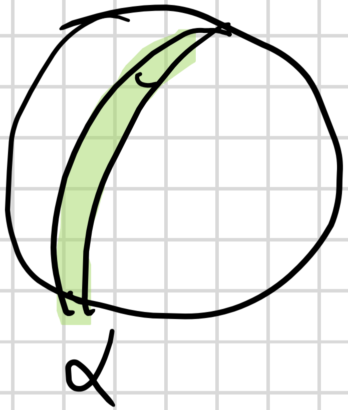


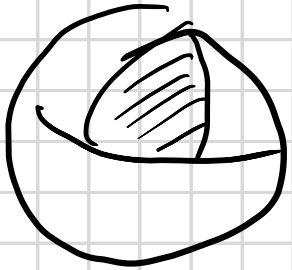
FIGURE 16.6. Generalized Π-congruences between a square and a circle.

② Completism holds for $\mathbb{S}^3$, $\mathbb{H}^3$

lune



$\mathbb{S}^2$



Theorem 41.1 (Girard's formula). *Let T be a spherical triangle with angles α, β and γ. Then* $\operatorname{area}(T) = \alpha + \beta + \gamma - \pi$.