HOME ASSIGNMENT 1 (MATH 285N, SPRING 2010)

1. Let $Q \subset \mathbb{R}^2$ be a convex polygon. A *qunti-partition* of Q is a point $z \in Q$ and five rays starting at z, which dives $\mathbb{R}^2$ into five equal cones, and divides Q into five polygons of equal area. Prove or disprove: every Q as above has a qunti-partition.

2. Let $Q_1, \ldots, Q_n \subset \mathbb{R}^2$ be (not necessarily convex) simple polygons in the plane (i.e. all Q_i are simply-connected). Suppose that every two of them intersect, and that all unions $Q_i \cup Q_j$ are connected, and all unions $Q_i \cup Q_j \cup Q_k$ are simply connected. Prove that the intersection of all Q_i is nonempty.

3. Let $Q \subset \mathbb{R}^2$ be a fixed convex n-gon (not necessarily regular), and let T_n be the set of triangulations of Q. We denote the vertices of Q by integers $i \in [n]$. Euler proved that $|T_n| = C_{n-2}$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ are *Catalan numbers.* For every vertex $v \in Q$ in a triangulation $\tau \in T_n$, denote by $\xi_\tau(v)$ the sum of areas of triangles in τ that contain v. Let $R_n \subset \mathbb{R}^n$ be a convex hull of all C_{n-2} functions $\xi_\tau : [n] \to \mathbb{R}^n$, defined as $v \to \xi_\tau(v)$, for all $\xi_\tau \in \mathbb{R}^n$.

a) Prove that all ξ_τ are vertices of R_n, i.e. that they lie in convex position.

b) Prove that the dimension of R_n is $(n-3)$.

c) Show that the flips of one diagonal in a triangulation correspond to edges of R_n. Conclude that R_n is simple.

4. Prove that a *permutohedron* P_n can be decomposed into n^{n-2} parallelepipeds with edges parallel to edges of P_n. For example, P_3 is a regular hexagon which can be decomposed into 3 such parallelograms (even in two different ways). Use this to compute the volume of P_n.

5. Let $P \subset \mathbb{R}^3$ be a simple convex polytope with only quadrilateral faces.

a) Prove that it has eight vertices.

b) Prove or disprove: P is combinatorially equivalent to a cube.

c) Prove that if seven vertices of P lie on a sphere, the so does the eighth vertex.

6. Let $P \subset \mathbb{R}^3$ be a convex polytope. Suppose we are allowed to cut P with a plane. Two parts are then separated, and each is then allowed to be separately cut with a new plane, etc. (think of chopping polytopes as if they were vegetables).

a) Prove that a cube can be cut into tetrahedral pieces with only four cuts.

b) Prove by an explicit construction that both icosahedron and dodecahedron can be cut into tetrahedral pieces with at most 100 cuts.

c) Prove that every convex polytope in $\mathbb{R}^3$ can be cut into tetrahedral pieces with finitely many cuts.