HOME ASSIGNMENT 3 (ENUMERATIVE COMBINATORICS, FALL 2010)

1. Let polytope $\mathbf{F}_n \subset \mathbb{R}^n$ be defined as a convex hull of Fibonacci $0-1$ sequences. Prove that $\operatorname{vol}\mathbf{F}_n = E_n/n!$, where E_n is Euler's number (the number of alternating permutations in S_n).

2. Let $f_n(t)$ be the inversion polynomial. Prove that $|f_n(-1)| = E_{n-1}$ in two different ways: using recurrence relations and an explicit involution.

3. A labeled tree is called *pretty* if every vertex is either greater or smaller than all its neighbors (this is somewhat similar to the notion of alternating permutations). Denote by $\mathcal{A}_n$ the hyperplane arrangement

$$\big\{x_i - x_j = 1 \mid 1 \le i < j \le n\big\} \subset \mathbb{R}^n .$$

Use the finite fields method to prove that the number of regions in the complement $\mathbb{R}^n \smallsetminus \mathcal{A}_n$ is the number of pretty trees on n vertices.

4. Denote by X_n the set of increasing tree such that every non-rooted vertex has an even number of sons. Denote by Y_n the set of set of increasing tress such that vertices at odd distance from the root have at most one son. Use exponential generating functions to prove that $|X_n|$ and $|Y_n|$ are Euler numbers.

5. Give a direct combinatorial proof that E_{2n+1} is divisible by 2^n.

6. Define *rooted* 2*-trees* to be a 3-uniform hypergraph on $[n]$ with $(n-2)$ hyperedges and *root* at edge (12), which satisfies the following condition:

for every vertex $v \in [n]$ there is a sequence of edges (i_1, j_1), ..., (i_ℓ, j_ℓ), $(1, 2)$, such that $(i_1 j_1 v)$ is a hyperedge, and every two subsequent edges in a sequence lie in some hyperedge.

For example, there is a unique 2-tree on [3], and five 2-trees on [4], given by $\{(123), (124)\}$, $\{(123), (234)\}$, $\{(123), (134)\}$, $\{(124), (134)\}$, and $\{(124), (234)\}$. Similarly, there are exactly 49 2-trees on [5]. Compute the number of 2-trees on n vertices.

7. Let $\mathcal{B}_n \subset \mathbb{R}^n$ be the following hyperplane arrangement:

$$\big\{x_i - x_j = 0,\, x_i - x_n = a \mid 1 \le i < j \le n, 1 \le a \le n - i\big\} .$$

Note that $\mathcal{B}_n$ has the same number of hyperplanes as the Shi arrangement $\mathcal{S}_n$. Prove that $\mathcal{B}_n$ and $\mathcal{S}_n$ have the same characteristic polynomial.

8. View the Vandermonde determinant product formula as a polynomial identity in n formal variables. Give a direct proof of this identity by an explicit involution.

9. Let $e_1, \ldots, e_n$ be a standard basis in $\mathbb{R}^n$. Let Q_n be the convex hull of O and all $e_i - e_j$, $1 \le i < j \le n$. Prove that $n! \cdot \operatorname{vol}\mathrm{Q}_n$ is the Catalan number.

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This HA is due Fri Nov 12, before class. No late solutions will be accepted as I plan to present some solutions in class. Note that some of these problems are easier than others.

Collaboration is allowed and even encouraged, but each student has to write her/his own solution. Moreover, each study group is limited to three, and I ask you to write your collaborators on the top of the first page.