HOME ASSIGNMENT 2 (ENUMERATIVE COMBINATORICS, FALL 2010)

1. Let T_n be the set of plane trees with n vertices, and let $\alpha, \beta : T_n \to \mathbb{Z}_+$ be two statistics: $\alpha(t)$ is the degree of the root, and $\beta(t)$ is the length of the 'left path' (root $\to$ oldest child $\to$ oldest child $\to \ldots$) Give a bijective proof that α and β are equidistributed.

2. *a*) Prove by induction that the number of binary trees with n vertices and k left edges is the *Narayana number* (see Stanley, Exc. 6.36).
b) Find a bijective proof that the number of Dyck paths of length $2n$ with $k+1$ local maxima is also the Narayana number.

3. Deduce Cayley's $x_1 \cdots x_n (x_1 + \ldots + x_n)^{n-2}$ formula directly from the matrix tree theorem. Use this idea to obtain the analogue of this formula for the complete bipartite graph $K_{m,n}$.

4. Use bijections and/or recurrence relations to prove that the following statistics are equidistributed:
(*i*) parking functions by $\binom{n}{2}$ minus the sum of the numbers
(*ii*) labeled trees by the number of inversions (an *inversion* is a pair $i < j$ such that j is on the way from i to the root at 1).

5. Define

$$f_G(z) = \sum_{\tau \in \mathcal{F}(G)} z^{c(\tau)-1}$$

where $\mathcal{F}(G)$ is the set of rooted forests in G, and $c(\tau)$ is the number of components of a forest τ.
a) Compute directly $f_G(z)$ when $G = K_n$.
b) Prove that $f_{\overline{G}}(-z) = (-1)^{n-1} f_G(z-n)$, where n is the number of vertices of G, and $\overline{G}$ is the complement graph.
c) Use part *b*) to compute the number of spanning trees in a complete tripartite graph $K_{p,q,r}$.
d) Use part *b*) to compute the number of spanning trees in K_n which contain a given forest with components of size $a_1, a_2 \ldots$

6. *a*) Points $1, \ldots, 2n$ are marked on a circle, and each is connected by a chord (straight interval) with some other marked point, such that the chords do not intersect. This makes a total of n non-intersecting chords, e.g. $(1-2, 3-4, 5-6)$ or $(1-4, 2-3, 5-6)$ for $n = 3$. Prove that the total number of such chord arrangements on $2n$ points is the Catalan number. For example, two cyclic translation of the first type and three of the second give all $C_3 = 5$ arrangements on 6 points.
b) Find the number of centrally-symmetric such chord arrangements.

7. Give a direct combinatorial proof of the Cayley-Hamilton theorem.

8. Define *labeled domino tilings* (l.d.t.) of a region Γ to be a domino tiling with non-repeated labels in $\{1, \ldots, \text{area}(\Gamma)/2\}$ on the dominos, so that the labels increase both horizontally and vertically. For example, 2×4 rectangle has 6 l.d.t., while only 5 unlabeled (since $F_4 = 5$).

a) Find the number of l.d.t. of a $2 \times n$ rectangle.

b) Find the number of l.d.t. of a $3 \times n$ rectangle.

9. Let $Z_2, \ldots, Z_n \in [n]$ be uniform independent r.v. In a complete graph K_n, consider the set T of $(n-1)$ edges $\big(i, \min\{i-1, Z_i\}\big)$, $1 < i \leq n$. Observe that T is a spanning tree in K_n. Relabel the vertices of K_n according to a uniform random permutation $\sigma \in S_n$. Prove that this results in a uniform random spanning tree in K_n.

10. Let X_n be a $2n \times 2n$ square, a, b, c, d are its corners (in cyclic order). Denote by $N_{pq}(\ell)$ the number of *walks* from p to q of length ℓ (walks in the graph are allowed to repeat vertices and edges, backtrack, etc.) For example, for $n = 1$, we have $N_{ab}(4) = 7$, while $N_{ac}(4) = 6$.

a) Prove that $N_{ab}(\ell) \geq N_{ac}(\ell)$ for all $\ell \geq 2n$.

b) Suppose now a, b, c, d are any four non-corner points on distinct sides of the square (in cyclic order). Prove:

$$\sum_{m=0}^{\ell} N_{ab}(m) \cdot N_{cd}(\ell - m) \geq \sum_{m=0}^{\ell} N_{ac}(m) \cdot N_{bd}(\ell - m)$$

for all $\ell \geq 2n$.

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This HA is due Mon Oct 25, before class. No late solutions will be accepted as I plan to present some solutions in class. Note that some of these problems are easier than others.

Collaboration is allowed and even encouraged, but each student has to write her/his own solution. Moreover, each study group is limited to three, and I ask you to write your collaborators on the top of the first page.