Problem 1. (15 points)

Solve the following LHRR: $a_{n+1} = 5a_n - 6a_{n-1}$, $a_1 = 9$, $a_2 = 20$.

Solution:

characteristic polynomial: $x^2 = 5x - 6 \Rightarrow x_1 = 2, x_2 = 3$

We have: $a_n = A \cdot 2^n + B \cdot 3^n$

Plug in $a_1$ and $a_2$ we get $A = \frac{7}{2}, B = \frac{2}{3}$
Problem 2. (20 points)

Find the number of subgraphs of $G$ isomorphic to $H$, where

a) $G = C_{12}$, $H = P_5$.
   Answer: 12

b) $G = K_{9,9}$, $H = C_6$.
   Answer: $3^2 \times 6$
   Hint: First choose $K_{3,3}$, then there are $6C_6$ in it

c) $G = K_{9,9}$, $H = C_7$.
   Answer: 0

d) $G = H_3$ (the 3-cube), $H = C_4$.
   Answer: 6 (only the 6 faces of the cube count)

e) $G = H_3$ (the 3-cube), $H = P_4$.
   Answer: 48 (Fix the central edge of $P_4$: There are 12 different central edges for $P_4$, each of them could induce 4 $P_4$)
Problem 3. (20 points)

For each of these sequences, either draw a simple graph with this score (degree sequence), or explain why there is no such graph.

a) (3,3,3,1,1,1)

Exist: Just draw a triangle and 3 extra vertices (one to one) connecting to each vertex of the triangle.

b) (4,4,4,1,1,1)

Not Exist: sum of degree is odd.

c) (5,5,5,3,2,1)

Not Exist: sum of degree is odd.

d) (4,4,...,4) ← 16 numbers.

Exist: You can draw 2 copies of $K_{4,4}$

e) (3,3,3,3,2,2,2)

Exist: You can draw a $C_8$ and add 2 chords
Problem 4. (15 points)

Decide whether the following pairs of graphs on 8 vertices are isomorphic or non-isomorphic.

a)

Isomorphic: for example: \(\{1,2,3,4,5,6,7,8\} \rightarrow \{a,b,f,e,d,c,g,h\}\)
You need to make sure that if 4 vertices construct a \(C_4\) then their mapped vertices also form a \(C_4\).

b)

Not isomorphic: the left graph does not have \(C_3\) as subgraph but the right one does.

Important: In case of isomorphism, you must present a bijection. In case non-isomorphism, you must present an argument.
Problem 5.  (30 points, 2 points each) TRUE or FALSE?

Circle correct answers with ink.  No explanation is required or will be considered.

T  F  (1) The hypercube graph $H_4$ contains an Eulerian cycle.
T  F  (2) The hypercube graph $H_4$ contains a Hamiltonian cycle.
T  F  (3) A subgraph of a connected graph is always connected.
T  F  (4) A subgraph of a disconnected graph is always disconnected.
T  F  (5) The sum of degrees of $K_{\ell,\ell}$ is $\ell^2$.
T  F  (6) The sum of degrees of a graph on $n \geq 3$ vertices is smaller than $n^2 - 1$.
T  F  (7) Computing the number of walks of given length in a graph can be done efficiently.
T  F  (8) Deciding whether a graph has Hamiltonian cycle can be done efficiently.
T  F  (9) The number of (shortest) grid walks $(0,0) \rightarrow (5,5)$ which do not go through $(1,3), (3,4), (4,1), (1,5)$ is $> 100$.
T  F  (10) Graph $H$ is a subgraph of $G$. Graph $H$ contains a Hamiltonian cycle. Then $G$ contains a Hamiltonian cycle.
T  F  (11) Graph $H$ is a subgraph of $G$. Graph $G$ contains a Hamiltonian cycle. Then $H$ contains a Hamiltonian cycle.
T  F  (12) Graph $K_{50,50}$ contains a subgraph isomorphic to $K_{10}$.
T  F  (13) Graph $K_{50,50}$ contains a subgraph isomorphic to $K_{10,10}$.
T  F  (14) The number of walks $1 \rightarrow 1$ of length $k$ in a graph $G$ on $n$ vertices can be computed via matrix $(A_G)^n$.
T  F  (15) Isomorphic graphs have the same number of Eulerian circuits.

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