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Total: (out of 100)
Problem 1. (20 points)

Compute the probability that 4-subset $A$ of $\{1, 2, \ldots, 10\}$ satisfies:

a) $A$ has no odd numbers,

b) $A$ has at least one number $\leq 3$,

c) $A$ contains 1 but not 7.

d) the smallest number in $A$ is divisible by 3

Solutions.

a) The 4-subsets of $\{1, \ldots, 10\}$ with no odd numbers are the 4-subsets of $\{2, 4, 6, 8, 10\}$ which has 5 elements. Therefore the probability is $\frac{\binom{5}{4}}{\binom{10}{4}} = \frac{1}{32}$.

b) The 4-subsets of $\{1, \ldots, 10\}$ with no number $\leq 3$ are the 4-subsets of $\{4, \ldots, 10\}$ which has 7 elements. Therefore the probability is $1 - \frac{\binom{3}{4}}{\binom{10}{4}} = \frac{5}{8}$.

c) The 4-subsets of $\{1, \ldots, 10\}$ containing 1 but not 7 can be expressed in one way as $\{1\} \cup S$ where $S$ is a 3-subset of $\{1, 2, 4, 5, 6, 8, 9, 10\}$ which has 8 elements. The probability must then be $\frac{\binom{8}{3}}{\binom{10}{4}} = \frac{4}{15}$.

d) The 4-subsets of $\{1, \ldots, 10\}$ with smallest number divisible by 3 must have smallest number 3 or 6. Here 9 is not an option because there are fewer than 4 numbers $\geq 9$ in the set. The 4-subsets of $\{1, \ldots, 10\}$ with smallest number 3 can be written in exactly one way as $\{3\} \cup S$ where $S$ is a 3-subset of $\{4, 5, 6, 7, 8, 9, 10\}$ which has 7 elements and each such $S$ makes $\{3\} \cup S$ a 4-subset with smallest number 3. There are then $\binom{7}{3}$ 4-subsets with smallest number 3. By similar reasoning there are $\binom{4}{3}$ subsets with smallest number 6. It follows that the probability is $\frac{\binom{7}{3} + \binom{4}{3}}{\binom{10}{4}} = \frac{13}{70}$.
Problem 2. (20 points)

Let $X = \mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of all non-negative integers. For each of the following functions $f : X \to X$ decide whether they are injective, surjective, bijective:

a) $f(x) = x + 1$

b) $f(x) = x^2 - 1$

c) $f(x) = 2x$

d) $f(x) = (x^2 + 2x)/(x + 2)$

Solutions.

a) That $f$ is injective but not surjective is a fundamental properties of $\mathbb{N}$. If $x, y \in \mathbb{N}$ and $f(x) = f(y)$, then $x + 1 = y + 1$ and so $x = y$. This shows $f$ is injective. Since $x + 1 \neq 0$ for all $x \in \mathbb{N}$, $f$ is not surjective and consequently is not bijective.

b) There is no $y \in X$ so that $y = 0^2 - 1$. Therefore it is not the case that $f$ is a function $X \to X$.

c) Suppose $x, y \in X$ and $f(x) = f(y)$. Then we would have $2x = 2y$ which gives us $x = y$. That $x = y$ follows from our assumptions shows that $f$ is injective. However $f$ is not surjective since $f(x) = 2x$ is an even integer whenever $x \in X = \mathbb{N}$ and so $f(x)$ cannot be an odd element of $\mathbb{N}$. For example, $f(x) \neq 1 \in X$ for every $x \in X$. Therefore $f$ is injective, but not surjective or bijective.

d) We have that for every $x \in \mathbb{N}$, $f(x) = (x^2 + 2x)/(x + 2) = x(x + 2)/(x + 2) = x$. Then $f$ is invertible, being the identity map on $X$ and therefore injective, surjective, and bijective.
Problem 3. (15 points)

Let \( a_n = 111 \cdots 1 \) (\( n \) ones). Suppose \( a_k \) is divisible by 97. Use induction to show that \( a_k \cdot n \equiv 0 \mod{97} \), for all \( n \geq 1 \).

Solution.

BASE: For \( n = 1 \), we have \( a_k \cdot 1 = a_k = a_k - 0 \) is divisible by 97. Therefore \( a_k = 0 \mod{97} \).

STEP: Suppose \( n \) was a particular natural number with the property that \( a_k \cdot n = 0 \mod{97} \). By the definition of the sequence \( a_1, a_2, \ldots \) and interpreting the base 10 expansion we have \( a_k \cdot (n+1) = a_k + 10^k a_k \cdot n \). By induction hypothesis and since \( 10^k \) is an integer, the product \( 10^k a_k \cdot n = 0 \mod{97} \). Recall that \( a_k = 0 \mod{97} \) (from the base case). Then the sum \( a_k \cdot (n+1) = 0 + 0 = 0 \mod{97} \). This completes the induction.
Problem 4.  (15 points)

Find closed formulas for the following sequences:

a) 4, 4, 6, 8, 12, 18, 28, 44, 70, 112, \ldots

b) \(a_1 = 1, \ a_{n+1} = a_n \cdot \binom{n+1}{2}\)

c) \(a_1 = 1, \ a_2 = 1, \ a_{n+1} = a_{n-1} - a_n \) for \(n \geq 2\).

Note: you can express \(a_n\) in terms of Fibonacci numbers \(F_n\).

Solutions. In each case, either of the following formulas is correct.

a) We have:
\[
a_n = 2F_n + 2 = 4 + 2(F_1 + F_2 + \ldots + F_{n-2})
\]

b) We have:
\[
a_n = \frac{(n-1)!^2 n}{2^{n-1}} = \frac{(n-1)! n!}{2^{n-1}}
\]

c) We have: 
\(a_n = (-1)^n F_{n-3}\) for \(n \geq 4\), and \(a_1 = a_2 = 1\) and \(a_3 = 0.\)
Problem 5. (30 points, 2 points each) TRUE or FALSE?

Circle correct answers with ink. No explanation is required or will be considered.

1. The number of functions from \( \{A, B, C, D\} \) to \( \{1, 2, 3\} \) is equal to \( 4^3 \).

2. The sequence \( 1, 3/2, 5/3, 7/6, 9/8, \ldots \) is increasing.

3. The sequence \( -1, -2, -3, -4, \ldots \) is non-increasing.

4. There are 4 anagrams of the word MAMA.

5. There are infinitely many Fibonacci numbers which are divisible by 3.

6. The number of permutations of \( \{1, 2, 3, 4, 5\} \) is smaller than 123.

7. The number of 3-permutations of \( \{1, 2, 3, 4, 5, 6\} \) is equal to \( \binom{6}{3} \).

8. The number of 3-subsets of \( \{1, 2, 3, 4\} \) is equal to 4.

9. The number of permutations of \( \{1, 2, \ldots, n\} \) which have \( n \) preceding \( n - 1 \) (not necessarily immediately) is equal to \( n!/2 \).

10. For every \( A, B \subset \{1, 2, \ldots, 12\} \) we have \( |A \cap B| < |A \cup B| \).

11. For all \( n \geq 1 \), we have
\[
\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.
\]

12. The number of grid walks from \( (0, 0) \) to \( (10, 10) \) going through \( (3, 7) \) is equal to \( \binom{10}{3}^2 \).

13. The number of grid walks from \( (0, 0) \) to \( (10, 10) \) avoiding \( (10, 0) \) and \( (0, 10) \) is equal to \( \frac{1}{2} \binom{20}{10} \).

14. The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

15. The following parabolas are drawn in the plane:
\[
y = x^2 - nx - n^3, \quad n = 1, \ldots, 12.
\]
Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.

Solutions. FFTFT TFTTF TFFFT