HOMEWORK 6 (MATH 61, SPRING 2017)

9.1

5. If m and n are both at least 2, then for $G = K_{m,n}$, with $V = V_1 \cup V_2$ the usual partition of the vertex set the induced subgraph of two vertices from V_1 and two vertices from V_2 is C_4. So if $m \geq 2$ and $n \geq 2$, $K_{m,n}$ is not a tree. If m or n is 0, then $K_{m,n}$ has no edges and so is a tree if and only if $K_{m,n}$ has one vertex or $m + n = 1$. If m or n is 1, then $K_{m,n}$ is a tree because it is a simple connected graph with as many vertices as one less than its number of edges. Summarizing the results, we find that $K_{m,n}$ is a tree exactly when one of m and n is 1.
31. A tree is a connected graph with no cycles. If K_5 or $K_{3,3}$ is homeomorphic to a subgraph of a tree then that subgraph has the same number of cycles as whichever of K_5 or $K_{3,3}$ it is homeomorphic to, which must be at least 1 (actually much more). This is impossible, so by Kuratowski's theorem all trees are planar. Another more elementary approach is to show by induction on number of vertices that every planar graph can be embedded in a plane with straight line segment edges.

9.2

16. 7. The parent of c is a. The parent of h is c respectively. 8. The ancestors of c consist only of a. The ancestors of j are a and d. 9. The children of d consist only of j. The children of e are k and l. 10. The descendants of c are g, h, i, m, and n. The descendants of e are k, l, q, and r. 11. There are no siblings of f. The siblings of h are g and i. 12. The terminal vertices are f, m, n, o, p, and r. 13. The internal vertices are a, b, c, d, e, g, j, l, and q. 14. The subtree rooted at j has vertices j, o, and p with an edge from j to o and an edge from j to p. 15. The subtree rooted at e has vertices e, k, l, q, r with edges from e to k, from e to l, from l to q, and from q to r.
23. One acyclic graph with 6 vertices and 4 edges is the disjoint union of P_5 and a single vertex.
24. No tree has all vertices of degree 2. For a graph with all vertices of degree 2 the sum of the degrees over all vertices would be both twice the number of vertices and twice the number of edges. So the number of edges must then be equal to the number of vertices which is not the case for trees ($v - 1 = e$ for trees).

9.3

8. There are many answer. For example there is the spanning tree with edges ab, bc, cd, de, bj, bk, ah, ai, al, af, and hg.

I. Since $K_{3,2}$ has five vertices, every spanning tree has four edges. Let $V_1 = \{a, b, c\}$ and $V_2 = \{d, e\}$ is the associated partition of the vertex set. Since every edge of $K_{3,2}$ is incident to exactly one element of V_1 and in a spanning tree, every vertex has degree at least one, if H is a spanning tree of $K_{3,2}$ then at exactly one element of V_1 has degree 2 in H with the other two degree 1. In fact any subgraph G of $K_{3,2}$ with degrees $(2, 1, 1)$ for the vertices in V_1 must be a tree since $K_{3,2}$ has no C_3 subgraphs and G has at most 3 vertices with degree at least 2. There are $3 \cdot 2^2 = 12$ spanning trees obtained from choosing a vertex of V_1 (to be degree 2) and one edge incident to each of the remaining elements of V_1.

II.

a. Take $G = P_7$.

b. The sum of all the degrees is odd. Therefore no such simple graph exists, let alone a tree.
c. For example, take the tree $V = a, b, c, d, e, f, g, h$ and $E = \{ab, ac, ad, be, cf, dg, eh\}$.
d. Take $K_{1,7}$.
e. Take $V = a, b, c, d, e, f, g, h$ and $E = \{ab, ac, ad, be, bf, cg, ch\}$.
f. The sum of all the degrees is 16 which means such a graph would need 16/2=8 edges. This cannot be a tree since we require exactly 7=8-1 edges for a tree on 8 edges.
g. Take the tree $V = a, b, c, d, e, f, g, h$ and $E = \{ab, ac, ad, be, bf, cg, dh\}$.

III. The adjacency matrix of $K_4 - e$ under a certain vertex order is $A := \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} K & J \\ J & 0 \end{pmatrix}$. where K and J and 0 are 2×2 matrices. Then $J^2 = 2J$ and $JK = KJ = J$. Then if $A^n = \begin{pmatrix} a_nJ + K^n & b_nJ \\ b_nJ & c_nJ \end{pmatrix}$, then $A^{n+1} = \begin{pmatrix} a_nJ + K^n & b_nJ \\ b_nJ & c_nJ \end{pmatrix} \begin{pmatrix} K & J \\ J & 0 \end{pmatrix} = \begin{pmatrix} (a_n + 2b_n)J + K^{n+1} & (2a_n + 1)J \\ (b_n + 2c_n)J & 2b_nJ \end{pmatrix}$. Since A is symmetric, so is A^{n+1} and so $b_n + 2c_n = 2a_n + 1$.

The above result is the induction step for the following: There exist sequences a_n, b_n, and c_n such that $a_1 = 0$, $b_1 = 1$, $c_1 = 0$, $a_{n+1} = a_n + 2b_n$, $b_{n+1} = 2a_n + 1 = b_n + 2c_n$, $c_{n+1} = 2b_n$, and $A^n = \begin{pmatrix} a_nJ + K^n & b_nJ \\ b_nJ & c_nJ \end{pmatrix}$ for all n. To solve this system we first observe $b_{n+2} = b_{n+1} + 2c_{n+1} = b_{n+1} + 4b_n$, $b_1 = 1$, and $b_2 = 1$. We define for convenience $b_0 = 0$ compatibly.

Therefore for some constants C and D we have $b_n = Cr_1^n + Dr_2^n$ for all n where $1 = Cr_1 + Dr_2$ and $0 = C + D$ and $r_1 = \frac{1+\sqrt{17}}{2}$, $r_2 = \frac{1-\sqrt{17}}{2}$. Solving this system gives $C = -D$ and since $r_1 - r_2 = \sqrt{17}$, $C = \frac{1}{\sqrt{17}}$. Thus $b_n = \frac{1}{\sqrt{17}}((\frac{1+\sqrt{17}}{2})^n - (\frac{1-\sqrt{17}}{2})^n)$. We can use this to compute $a_n = \frac{b_{n+1}-1}{2}$, $c_n = 2b_{n-1}$, and in particular a_{310}, b_{310}, and c_{310}.

The number of walks in $K_3 - e$ of length 310 from a vertex of degree 3 to itself is the upper left corner entry of A^{310} which in turn is the sum of a_n and the upper left corner entry of $K^{310} = (K^2)^{105} = I^{105} = I$ where I is the identity matrix. This number is $a_{310} + 1 = \frac{1}{2}(\frac{1}{\sqrt{17}}((\frac{1+\sqrt{17}}{2})^{311} - (\frac{1-\sqrt{17}}{2})^{311}) + 1)$.

The number of walks in $K_3 - e$ of length 310 from a vertex of degree 2 to itself is the lower right corner entry of A^{310} which is $c_{310} = \frac{2}{\sqrt{17}}((\frac{1+\sqrt{17}}{2})^{309} - (\frac{1-\sqrt{17}}{2})^{309})$.

IV. The weights of edges of the graph are all distinct so we will call an edge by its weight. The minimal spanning tree consists of the edges 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 23, 24, 27, 29, 32, 42, 44, 45, 46, 49, 51, 52, 54, 55, 56, 57, 60, 61, 62, 71, 72, 75, 76, 84, 86, 87, 88, 90, 121, and 166. The maximal spanning tree consists of the edges 9, 17, 19, 20, 21, 22, 23, 27, 30, 33, 34, 35, 44, 45, 46, 51, 52, 62, 73, 77, 78, 79, 82, 83, 86, 100, 101, 110, 123, 131, 141, 152, 159, 166, 174, 188, 199, 211, 311, 417, 518, 611, 711, 712, 844, 911, 922