HOMEWORK 4 (MATH 61, SPRING 2015)

NOTE: Everywhere below, we use book notation:

$$C(n,k) = \binom{n}{k}$$

6.3

2. $\frac{6!}{2!} = 360$.
3. $\frac{12!}{4!2!}$.

6.7

15. From the Binomial Theorem
$$(a+b)^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k,$$
let $a = 1$ and $b = -1$. We have
$$0 = (1-1)^n = \sum_{k=0}^{n} C(n,k)1^{n-k}(-1)^k = \sum_{k=0}^{n} (-1)^k C(n,k).$$
20. From Example 6.7.8 with $k = 2$, we have
$$\sum_{i=2}^{n} C(i,2) = C(n+1,3).$$
Since $C(i,2) = \frac{i(i-1)}{2}$,
$$1\cdot 2 + 2\cdot 3 + (n-1)n = 2\cdot \sum_{i=2}^{n} C(i,2) = 2C(n+1,3) = \frac{(n+1)n(n-1)}{3}.$$

7.2

16. If $a_n = 7a_{n-1} - 10a_{n-2}$; $a_0 = 5$, and $a_1 = 16$, then the characteristic equation $\lambda^2 = 7\lambda - 10$ can be rewritten $(\lambda-5)(\lambda-2) = 0$ and so has two single roots $\lambda = 2, 5$. Now there are constants c_1 and c_2 so that $a_n = c_1 2^n + c_2 5^n$. So we obtain a linear system consisting of $5 = c_1 + c_2$ and $16 = 2c_1 + 5c_2$. Equivalently, $5 = c_1 + c_2$ and $6 = 3c_2$, which has a unique solution $c_1 = 3$, $c_2 = 2$. So $a_n = 3\cdot 2^n + 2\cdot 5^n$ for every natural number $n \geq 0$.
17. If $a_n = 2a_{n-1} + 8a_{n-2}$; $a_0 = 4$, and $a_1 = 10$, then the characteristic equation $\lambda^2 = 2\lambda + 8$ can be rewritten $(\lambda-4)(\lambda+2) = 0$ and so has two single roots $\lambda = -2, 4$. Now there are constants c_1 and c_2 so that $a_n = c_1(-2)^n + c_2 4^n$. So we obtain a linear system consisting of $4 = c_1 + c_2$ and $10 = -2c_1 + 4c_2$. Equivalently, $4 = c_1 + c_2$ and $18 = 6c_2$, which has a unique solution $c_1 = 1$, $c_2 = 3$. So $a_n = (-2)^n + 3\cdot 4^n$ for every natural number $n \geq 0$.
22. If $9a_n = 6a_{n-1} - a_{n-2}$; $a_0 = 6$, and $a_1 = 5$, then the characteristic equation $9\lambda^2 = 6\lambda - 1$ can be rewritten $(3\lambda - 1)^2 = 0$ and so has a double root $\lambda = 1/3$. Now there are constants c_1 and c_2 so that $a_n = c_1(1/3)^n + c_2 n(1/3)^n$. So we obtain a linear system consisting of $6 = c_1 + 0c_2$ and $5 = (1/3)c_1 + (1/3)c_2$. We get $c_1 = 6$ and $c_2 = 9$. So $a_n = 6(1/3)^n + 9n(1/3)^n$ for every natural number $n \geq 0$.

I. Find the number of anagrams of MISSISSIPPI

a) $C(10,4)C(6,3)C(3,2) = \frac{10!}{4!3!2!}$.
b) $C(9,4)C(5,2)C(3,2) = \frac{9!}{4!2!2!}$.
c) $C(10,4)C(6,4)C(2,1) = \frac{10!}{4!4!2!}$.
d) $C(8,4)C(4,2)C(2,1) = \frac{8!}{4!2!}$.
e) $C(9,4)C(5,2)C(3,2) = \frac{9!}{4!2!2!}$.
f) $C(11,6)C(5,4) = \frac{11!}{6!4!}$.
g) $C(11,8)C(3,2) = \frac{11!}{8!2!}$.
h) 0.
i) $2C(10,4)C(6,4)C(2,1) - C(9,4)C(5,4) = 2 \cdot \frac{10!}{4!4!} - \frac{9!}{4!4!}$.
j) $2C(10,4)C(6,3)C(3,2) - C(9,4)C(5,2)C(3,2) = 2 \cdot \frac{10!}{4!3!2!} - \frac{9!}{4!2!2!}$.

II. On a grid, let $A = (0,0)$, $B = (10,12)$, $C = (3,2)$, $D = (6,9)$, $E = (9,7)$. The number of (shortest) grid walks $A \to B$ which

a) go through C are $C(5,3)C(17,7)$.
b) don't go through D are $C(22,10) - C(15,6)C(7,4)$.
c) go through C but not D are $C(5,3)C(17,7) - C(5,3)C(10,3)C(7,4)$.
d) go through E but not C are $C(16,9)C(6,1) - C(5,3)C(11,6)C(6,1)$.
e) go through C and E are $C(5,3)C(11,6)C(6,1)$.
f) go through either D or E are $C(15,6)C(7,4) + C(16,9)C(6,1)$.
g) go through D and E are 0.
h) go through neither C nor D nor E $C(22,10) - C(5,3)C(17,7) - C(15,6)C(7,4) - C(16,9)C(6,1) + C(5,3)C(10,3)C(7,4) + C(5,3)C(11,6)C(6,1)$.

III. Prove the following results about Fibonacci numbers F_n :

a) there are infinitely many n such that $F_n = 0 \mod 7$

For any $n > 8$, $F_n = F_{n-1} + F_{n-2} = 2F_{n-2} + F_{n-3} = 3F_{n-3} + 2F_{n-4} = 5F_{n-4} + 3F_{n-5} = 8F_{n-5} + 5F_{n-6} = 13F_{n-6} + 8F_{n-7} = 21F_{n-7} + 13F_{n-8} = 13F_{n-8}(mod 7)$. Since $F_8 = 21 = 0 \mod 7$, $F_n = 0 \mod 7$ for any $n = 0 \mod 7$. Therefore, there are infinitely many n such that $F_n = 0 \mod 7$.

b) there are infinitely many n such that F_n begins with 1.

Let k be a positive integer. Let F_n be the largest Fibonacci number such that $F_n < 2 \cdot 10^k$. Then $F_{n+1} \geq 10^k$ and $F_{n+1} = F_n + F_{n-1} < 2F_n < 2 \cdot 10^k$. Therefore, F_{n+1} is a $(k+1)$-digit number that begins with 2. Since we can find such number for every $k > 0$, there are infinitely many n such that F_n begins with 1.

IV. Draw two non-isomorphic graphs with scores (degree sequences)

a) (3,3,3,3,5,5,6,6,6) [Answers will vary. One strategy is to form a complete graph with 5 vertices and draw edges between those 5 vertices and 4 remaining vertices as well as two edges between two pairs of the four vertices to result in graphs with the given score. To ensure that they are non-isomorphic one can for instance choose graphs so that the degree 5 vertices are incident to the same degree 3 vertex or not.]

b) (3,3,3,5,5,5,6,7,7) [Answers will vary. The strategy outlined above can be modified to begin with a complete graph on 6 vertices. This complete graph can be extended so that a triangle is formed by degree 3, 6, and 7 vertices or so that such a triangle is not formed.]

V. We will show $a_n = F_{2n}$ by induction. In the base case, we have $a_1 = F_1 = 1 = F_2 = F_{2\cdot 1}$. Assuming n is a natural number so that $a_n = F_{2n}$, we have $a_{n+1} = a_n + F_{2(n+1)-1} = F_{2n} + F_{2n+1} = F_{2n+2} = F_{2(n+1)}$ using the induction hypothesis in the second equality of the chain.

VI. Using the closed formula $F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$, and letting $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$ we have for $n > 1$,

$$\begin{aligned}
5F_{n+1}F_{n-1} &= (\phi^{n+1} - \psi^{n+1})(\phi^{n-1} - \psi^{n-1}) \\
&= \phi^{2n} - (\phi\psi)^{n-1}(\phi^2 + \psi^2) + \psi^{2n} \\
&= (\phi^n - \psi^n)^2 + 2(\phi\psi)^n - (\phi\psi)^{n-1}(\phi^2 + \psi^2) \\
&= 5(\frac{1}{\sqrt{5}}\phi^n - \frac{1}{\sqrt{5}}\psi^n)^2 - (\phi\psi)^{n-1}(\phi - \psi)^2 \\
&= 5F_n^2 - (-1)^{n-1}(\sqrt{5})^2 \\
&= 5F_n^2 + 5(-1)^n.
\end{aligned}$$

Above we used the identities $\phi\psi = -1$ and $\phi - \psi = \sqrt{5}$. By dividing both sides by 5 we obtain $F_{n+1}F_{n-1} = F_n^2 + (-1)^n$ as needed.

VII. We will solve $a_{n+1} = 4a_n - 3a_{n-1}$ for three sets of initial conditions. The polynomial $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$ has two single roots at $\lambda = 1, 3$ so solutions to the recurrence are of the form $a_n = c_1 + c_2 3^n$ for all n for some constants c_1, c_2.

i) Under the initial conditions $a_1 = 1$ and $a_2 = 3$, we have $1 = c_1 + 3c_2$ and $3 = c_1 + 9c_2$. Solving this system gives $c_2 = 1/3$ and $c_1 = 0$ so that $a_n = 3^{n-1}$ for all $n \geq 1$. Let us verify this by induction. In the base cases we have $a_1 = 1 = 3^0 = 3^{1-1}$ and $a_2 = 3 = 3^1 = 3^{2-1}$. Assuming n is a natural number such that both $a_n = 3^{n-1}$ and $a_{n-1} = 3^{(n-1)-1}$, we have $a_{n+1} = 4a_n - 3a_{n-1} = 4 \cdot 3^{n-1} - 3 \cdot 3^{(n-1)-1} = 4 \cdot 3^{n-1} - 3^{n-1} = 3 \cdot 3^{n-1} = 3^{(n+1)-1}$, completing the induction.

ii) Under the initial conditions $a_1 = 5$ and $a_2 = 5$, we have $5 = c_1 + 3c_2$ and $5 = c_1 + 9c_2$. Solving this system gives $c_2 = 0$ and $c_1 = 5$ so that $a_n = 5$ for all $n \geq 1$. Let us verify this by induction. In the base cases we have $a_1 = 5$ and $a_2 = 5$ by our initial conditions. Assuming n is a natural number such that both $a_n = 5$ and $a_{n-1} = 5$, we have $a_{n+1} = 4a_n - 3a_{n-1} = 4 \cdot 5 - 3 \cdot 5 = 5$, completing the induction.

iii) Under the initial conditions $a_1 = 2$ and $a_2 = 4$, we have $2 = c_1 + 3c_2$ and $4 = c_1 + 9c_2$. Solving this system gives $c_2 = 1/3$ and $c_1 = 1$ so that $a_n = 1 + 3^{n-1}$ for all $n \geq 1$. Let us verify this by induction. In the base cases we have $a_1 = 2 = 1 + 1 = 1 + 3^0 = 1 + 3^{1-1}$ and $a_2 = 4 = 1 + 3 = 1 + 3^{2-1}$ by our initial conditions. Assuming n is a natural number such that both $a_n = 1 + 3^{n-1}$ and $a_{n-1} = 1 + 3^{(n-1)-1}$, we have $a_{n+1} = 4a_n - 3a_{n-1} = 4(1 + 3^{n-1}) - 3(1 + 3^{(n-1)-1}) = 4 + 4 \cdot 3^{n-1} - 3 - 3^{n-1} = 1 + (4 - 1)3^{n-1} = 1 + 3^{(n+1)-1}$, completing the induction.