Warning: everywhere below we follow book notation:

\[ P(n, k) = \frac{n!}{(n-k)!} \quad \text{and} \quad C(n, k) = \binom{n}{k}. \]

6.2

6. \( P(11, 5) = \frac{11!}{6!} \).

8. \( P(12, 4) = \frac{12!}{8!} \).

29. \( C(12, 4) = \frac{12!}{8! \cdot 4!} \).

34. \( C(6, 3) \cdot C(7, 4) \).

35. (# of Total Committees) - (# of All Male Committees) = \( C(13, 4) - C(6, 4) \).

37. (# of Total Committees) - (# of All Male Committees) - (# of All Female Committees) = \( C(13, 4) - C(6, 4) - C(7, 4) \).

6.7

2. \((2c - 3d)^5 = \sum_{k=0}^{5} C(5, k)2^k(-3)^{5-k}c^k d^{5-k} = 32c^5 - 240c^4d + 720c^3d^2 - 1080c^2d^3 + 810cd^4 - 243d^5 \)

4. \( C(12, 6)2^6(-1)^6 = \frac{12!2^6}{(6!)^2} \)

5. \( C(10, 5)C(5, 3) = \frac{10!}{5!2!3!} \)

I.

Before we begin, here is a counting principle (CP) you can use repeatedly throughout this problem.

Say you have distinct natural numbers \( 1 \leq i_1 < i_2 < \ldots < i_k \leq n \) and and ordered \( k \)-tuple of distinct natural numbers between 1 and \( k \), \((b_1, \ldots, b_k)\). How many permutations of \( \{1, \ldots, n\} \) are there with \( a_{i_j} = b_j \) for each \( j = 1, \ldots, k \)? There are \( (n-k)! \) of them.

Here is why: There is a bijection between the set \( S \) of permutations of the elements of \( \{1, \ldots, n\} \) \( \backslash \{b_1, \ldots, b_k\} \) (the set of natural numbers between 1 and \( n \) that are not equal to any of the \( b_i \)) and the set \( T \) of permutations we are counting. The bijection \( f : T \rightarrow S \) takes a permutation in \( T \) and deletes \( b_1, \ldots, b_k \).

Here is a detailed explanation of why \( f \) is bijective. First, let us start with why \( f \) is injective. Take two permutations in \( T \) that become the same when \( b_1, \ldots, b_k \) are deleted. They (the two permutations in \( T \)) must agree in every position besides \( i_1, \ldots, i_k \) because for elements of \( T \), \( b_1, \ldots, b_k \) occur in the \( i_1, \ldots, i_k \) positions. However they also agree in the \( i_1, \ldots, i_k \) positions by the definition of \( T \) (permutations with \( b_j \) in the \( i_j \) position). Therefore, they agree at every position and are equal (the same) as permutations. To see \( f \) is surjective begin with an element of \( S \), choose one. Create a permutation by drawing \( n \) blanks (ordered) and writing \( b_j \) on the \( i_j \)-th blank. Fill the remaining \( n-k \) remaining blanks with your chosen element of \( S \); you will get an element of \( T \). When you apply \( f \) to this element of \( T \), all the \( b_j \) are deleted and you are left with your chosen element of \( S \). Since this argument works for any choice of an element of \( S \), \( f \) is surjective.

a) Since \( a_1a_n = 6 \), as \( 6 = 1 \cdot 6 = 6 \cdot 1 = 2 \cdot 3 = 3 \cdot 2 \) are the only factorizations of 6 into two distinct natural numbers (between 1 and \( n = 12 \geq 6 \)), we are counting the number of permutations such that the ordered pair \((a_1, a_n)\) is either \((1, 6)\), \((6, 1)\), \((2, 3)\), or \((3, 2)\) and no two cases can simultaneously happen.
So by the addition principle, the number of permutations with \( a_1a_n = 6 \) is the sum of the number of permutations in each of the four cases. Using CP, each case has \((n - 2)!\) elements. Therefore there are \(4((n - 2)!) = 4(10!)\) permutations with \( a_1a_n = 6 \). Answer: \(4(10!)\).

b) We have \( a_1 - a_n = n - 1 \) if and only if \( a_1 = n \) and \( a_n = 1 \). Why? First check that \( a_1 = n \) and \( a_n = 1 \) ensures that \( a_1 - a_n = n - 1 \) (just substitute and see the equation is true). Any permutation has \( a_1 - a_n = a_1 + (-a_n) \leq n + (-a_n) \leq n + (-1) = n - 1 \). So when \( a_1 - a_n = n - 1 \) the chain of inequalities loops back to the beginning. Since \( \leq \) is antisymmetric we get that \( a_1 - a_n = n - a_n = n - 1 \) with the first equality giving \( a_1 = n \) and the second giving \( a_n = 1 \). Now by CP, there are \((n - 2)! = 10!\) such permutations. Answer: \(10!\).

c) Let \( S \) be the set of ordered pairs \((b, c)\) of distinct \((b \neq c)\) integers in \([1, \ldots, n]\) such that \( b + c = n + 2 \). Then \( S \) is the set resulting from removing the pair \((\frac{n + 2}{2}, \frac{n + 2}{2} + 1)\) from \((2, 3, n - 1, 4, n - 2, \ldots, (n, 1))\) \((n \text{ even})\). Then \( S \) has \(n - 2\) elements. \((\text{If } n \text{ were odd, then there would have been } n - 1 \text{ elements because we would not need to remove } (\frac{n + 2}{2}, \frac{n + 2}{2} + 1)!\)

The number of permutations with \( a_1 + a_n = n + 2 \) by the addition principle is the sum of number of permutations where \((a_1, a_n) = (b, c)\) where \((b, c)\) varies over the elements of \( S \). By CP, whenever \((b, c) \in S\), the number of permutations where \((a_1, a_n) = (b, c)\) is \((n - 2)!\). Therefore the total is the sum of \(n - 2\) copies of \((n - 2)!\). Answer: \(10(10!)\). Answer: \(10(10!)\).

d) This is a special case of CP. We have \((n - 2)! = 10!\) permutations with \( a_1 = 1 \) and \( a_n = n \). Answer: \(10!\).

e) Let \( S \) be the set of ordered pairs \((b, c)\) of distinct integers in \([1, \ldots, n]\) such that \( b = 2 \) or \( c = 3 \). There are \(n - 1\) elements of \( S \) of the form \((2, c)\) and \(n - 1\) elements of \( S \) of the form \((b, 3)\) \([2, 2), (3, 3) \notin S]\). All elements of \( S \) fall into one of these two categories and exactly one element, \((2, 3)\), falls into both. Then \( S \) has \((n - 1) + (n - 1) - 1 = 2n - 3\) elements.

By the strategy used in part c the number of permutations with \( a_1 = 2 \) or \( a_2 = 3 \) is \((2n - 3)((n - 2)!) = 21(10!)\). Answer 21(10!).

f) Let \( S \) be the set of ordered pairs \((b, c)\) of distinct integers in \([1, \ldots, n]\) such that \( b \leq 3 \) or \( c \geq 3 \). The number of elements of \( S \) with \( b \leq 3 \) is \(3(n - 1)\) \([\text{since } (1, 1), (2, 2), \text{ and } (3, 3) \text{ are not in } S]\) and similarly the number of elements of \( S \) with \( c \geq 3 \) is \((n - 2)(n - 1)\). The number of elements \((b, c)\) of \( S \) with both \( b \leq 3 \) and \( c \geq 3 \) is \(3(n - 2) - 1\) \([\text{since } (3, 3) \notin S]\). Therefore the total number of elements of \( S \) is \(3(n - 1) + (n - 2)(n - 1) - (3(n - 2) - 1) = n^2 - 3n + 4\).

Therefore the number of permutations with \( a_1 \leq 3 \) or \( a_2 \geq 3 \) is \((n^2 - 3n + 4)((n - 2)!) = 112(10!)\). Answer: 112(10!).

g) Take \( S \) to be the set of ordered pairs \((b, c, d)\) of pairwise distinct \((b \neq c, c \neq d, \text{ and } b \neq d)\) integers in \([1, \ldots, n]\) such that \( b = 2 \) or \( c = 3 \) or \( d = 4 \). There are \((n - 1)(n - 2)\) elements of the form \((2, c, d)\). Similarly, there are \((n - 1)(n - 2)\) elements of the form \((b, 3, d)\), and \((n - 1)(n - 2)\) of the form \((b, c, 4)\). There are \((n - 2)\) elements of the form \((2, 3, d)\) and same for \((2, c, 4)\) and \((b, 3, 4)\). Of course, there is only one element of \( S \) of the form \((2, 3, 4)\). By inclusion-exclusion principle there are \(((n - 1)(n - 2) + (n - 1)(n - 2) + (n - 1)(n - 2)) - ((n - 2) + (n - 2) + (n - 2)) + 1 = 3n^2 - 12n + 13\) elements in \( S \). Therefore the number of permutations with \( a_1 = 2, a_2 = 3, \text{ or } a_3 = 4 \) is \((3n^2 - 12n + 13)((n - 3)!) = 301(9!)\). Answer 301(9!).

II.

a) We need \(k - 2\) other elements from \([2, \ldots, n - 1]\). There are \(C(n - 2, k - 2)\) ways to choose them. Answer: \(C(10, 2)\).

b) We need \(k - 1\) elements from \([2, \ldots, n - 1]\). There are \(C(n - 2, k - 1)\) ways to choose them. Answer: \(C(10, 3)\).

c) By the inclusion-exclusion principle, \(C(n - 1, k - 1) + C(n - 1, k - 1) - C(n - 2, k - 2)\). Answer: \(2 \times C(11, 3) - C(10, 2)\).

d) There are \(C(n - 5, k)\) \(k\)-subsets which do not contain at least one integer \(\leq 5\). We subtract this from the total number of \(k\)-subsets. Answer: \(C(12, 4) - C(7, 4)\).
e) The number of $k$-subsets not containing an integer $\leq 3$ is $\binom{n-3}{k}$. The number of $k$-subsets not missing any integer $\geq 10$ is $\binom{n-(n-9)}{k-(n-9)} = \binom{9}{k-(n-9)}$. Therefore the number of $k$-subsets both not missing any integer $\geq 10$ and not containing an integer $\leq 3$ is $\binom{6}{k-(n-9)}$.

Therefore the number of $k$-subsets containing an integer $\leq 3$ and missing at least one integer $\geq 10$ is $\binom{n}{k} - \binom{n-3}{k} - \binom{9}{k-(n-9)} + \binom{6}{k-(n-9)}$. Answer: $\binom{12}{4} - \binom{9}{4} - 3$.

f) We subtract subsets with fewer than 2 numbers less than or equal to 6 from the total number of $k$-subsets. The number of $k$-subsets with exactly one number less than or equal to 6 are $6\binom{n-6}{k-1}$ by the multiplication principle. The number of $k$-subsets with zero numbers less than or equal to 6 are $\binom{n-6}{k}$. So $\binom{n}{k} - 6\binom{n-6}{k-1} - \binom{n-6}{k}$ subsets have at least two numbers less than or equal to 6. Answer: $\binom{12}{4} - 6\binom{6}{3} - \binom{6}{4}$

g) There are $\frac{n}{2}$ even integers in $\{1, \ldots, n\}$ (since $n$ is even - otherwise we would round down) to choose from. Answer: $\binom{6}{4}$.

III.

Because $f$ maps a finite set to itself, injection, surjection, and bijection are all the same.

a) Assume $f(x) = f(y)$. Then $x+1 = y+1 \mod 12$ and $x = y \mod 12$. Therefore since $0 \leq x, y < 12$, $x = y$. Answer: Injection, surjection, bijection.

b) Assume $f(x) = f(y)$. Then $5x = 5y \mod 12$ and so 12 divides $5(x-y)$. Since 5 and 12 have no common factors, by the fundamental theorem of arithmetic, 12 must divide $x-y$. Thus $x = y \mod 12$. We conclude like in part a that $x = y$. Answer: Injection, surjection, bijection.

c) It is not injective because $0^2 = 6^2 \mod 12$. Answer: Neither.

d) It is not surjective because $0^3 = 6^3 \mod 12$. Answer: Neither.

e) It is not surjective because since powers of 5 are always odd and thus not divisible by 12. That is there is no $x$ for which $5^x = 0 \mod 12$ which is equivalent to $f(x) = 0$. Answer: Neither.

f) Observe that $f(2) = 3 = f(4)$. Answer: Neither.

g) Observe that $f(3) = 0 = f(6)$. Answer: Neither.