MATH 61 (SPRING 2017): HOMEWORK 3

Warning: everywhere below we follow book notation:

$$P(n,k) = \frac{n!}{(n-k)!} \qquad \text{and} \qquad C(n,k) = \binom{n}{k}.$$

6.2

6. $P(11,5) = \frac{11!}{6!}$.

8. $P(12,4) = \frac{12!}{8!}$.

29. $C(12,4) = \frac{12!}{8! \cdot 4!}$.

34. $C(6,3) \cdot C(7,4)$

35. (# of Total Committees) - (# of All Male Committees) $= C(13,4) - C(6,4)$.

37. (# of Total Committees) - (# of All Male Committees) - (# of All Female Committees) $= C(13,4) - C(6,4) - C(7,4)$.

6.7

2. $(2c-3d)^5 = \sum_{k=0}^{5} C(5,k)2^k(-3)^{5-k}c^k d^{5-k} = 32c^5 - 240c^4d + 720c^3d^2 - 1080c^2d^3 + 810cd^4 - 243d^5$

4. $C(12,6)2^6(-1)^6 = \frac{12! \cdot 2^6}{(6!)^2}$

5. $C(10,5)C(5,3) = \frac{10!}{5! \cdot 2! \cdot 3!}$

I.

Before we begin, here is a counting principle (CP) you can use repeatedly throughout this problem.

Say you have distinct natural numbers $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and and ordered k-tuple of distinct natural numbers between 1 and k, $(b_1, \ldots, b_k)$. How many permutations of $\{1, \ldots, n\}$ are there with $a_{i_j} = b_j$ for each $j = 1, \ldots, k$? There are $(n-k)!$ of them.

Here is why: There is a bijection between the set S of permutations of the elements of $\{1, ..., n\} \setminus \{b_1, \ldots, b_k\}$ (the set of natural numbers between 1 and n that are not equal to any of the b_i) and the set T of permutations we are counting. The bijection $f : T \to S$ takes a permutation in T and deletes $b_1, ..., b_k$.

Here is a detailed explanation of why f is bijective. First, let us start with why f is injective. Take two permutations in T that become the same when $b_1, ..., b_k$ are deleted. They (the two permutations in T) must agree in every position besides $i_1, ..., i_k$ because for elements of T, $b_1, ..., b_k$ occur in the $i_1, ..., i_k$ positions. However they also agree in the $i_1, ..., i_k$ positions by the definition of T (permutations with b_j in the i_j position). Therefore, they agree at every position and are equal (the same) as permutations. To see f is surjective begin with an element of S, choose one. Create a permutation by drawing n blanks (ordered) and writing b_j on the i_j-th blank. Fill the remaining $n-k$ remaining blanks with your chosen element of S; you will get an element of T. When you apply f to this element of T, all the b_j are deleted and you are left with your chosen element of S. Since this argument works for any choice of an element of S, f is surjective.

a) Since $a_1 a_n = 6$, as $6 = 1 \cdot 6 = 6 \cdot 1 = 2 \cdot 3 = 3 \cdot 2$ are the only factorizations of 6 into two distinct natural numbers (between 1 and $n = 12 \geq 6$), we are counting the number of permutations such that the ordered pair (a_1, a_n) is either $(1,6)$, $(6,1)$, $(2,3)$, or $(3,2)$ and no two cases can simultaneously happen.

So by the addition principle, the number of permutations with $a_1a_n = 6$ is the sum of the number of permutations in each of the four cases. Using CP, each case has $(n-2)!$ elements. Therefore there are $4((n-2)!) = 4(10!)$ permutations with $a_1a_n = 6$. Answer: $4(10!)$.

b) We have $a_1 - a_n = n-1$ if and only if $a_1 = n$ and $a_n = 1$. Why? First check that $a_1 = n$ and $a_n = 1$ ensures that $a_1 - a_n = n-1$ (just substitute and see the equation is true). Any permutation has $a_1 - a_n = a_1 + (-a_n) \le n + (-a_n) \le n + (-1) = n-1$. So when $a_1 - a_n = n-1$ the chain of inequalities loops back to the beginning. Since $\le$ is antisymmetric we get that $a_1 - a_n = n - a_n = n-1$ with the first equality giving $a_1 = n$ and the second giving $a_n = 1$. Now by CP, there are $(n-2)! = 10!$ such permutations. Answer: $10!$.

c) Let S be the set of ordered pairs (b, c) of distinct $(b \neq c)$ integers in $\{1, \ldots, n\}$ such that $b+c = n+2$. Then S is the set resulting from removing the pair $(\frac{n}{2}+1, \frac{n}{2}+1)$ from $\{(2, n), (3, n-1), (4, n-2), ..., (n, 2)\}$ (n is even). Then S has $n-2$ elements. (If n were odd, then there would have been $n-1$ elements because we would not need to remove $(\frac{n}{2}+1, \frac{n}{2}+1)$!)

The number of permutations with $a_1 + a_n = n+2$ by the addition principle is the sum of number of permutations where $(a_1, a_n) = (b, c)$ where (b, c) varies over the elements of S. By CP, whenever $(b, c) \in S$, the number of permutations where $(a_1, a_n) = (b, c)$ is $(n-2)!$. Therefore the total is the sum of $n-2$ copies of $(n-2)!$ or $(n-2)(n-2)! = 10(10!)$. Answer: $10(10!)$.

d) This is a special case of CP. We have $(n-2)! = 10!$ permutations with $a_1 = 1$ and $a_n = n$. Answer: $10!$.

e) Let S be the set of ordered pairs (b, c) of distinct integers in $\{1, \ldots, n\}$ such that $b = 2$ or $c = 3$. There are $n-1$ elements of S of the form $(2, c)$ and $n-1$ elements of S of the form $(b, 3)$ $[(2, 2), (3, 3) \notin S]$. All elements of S fall into one of these two categories and exactly one element, $(2, 3)$, falls into both. Then S has $(n-1) + (n-1) - 1 = 2n-3$ elements.

By the strategy used in part c the number of permutations with $a_1 = 2$ or $a_2 = 3$ is $(2n-3)((n-2)!) = 21(10!)$. Answer $21(10!)$.

f) Let S be the set of ordered pairs (b, c) of distinct integers in $\{1, \ldots, n\}$ such that $b \le 3$ or $c \ge 3$. The number of elements of S with $b \le 3$ is $3(n-1)$ [since $(1, 1)$, $(2, 2)$, and $(3, 3)$ are not in S] and similarly the number of elements of S with $c \ge 3$ is $(n-2)(n-1)$. The number of elements (b, c) of S with both $b \le 3$ and $c \ge 3$ is $3(n-2) - 1$ [since $(3, 3) \notin S$]. Therefore the total number of elements of S is $3(n-1) + (n-2)(n-1) - (3(n-2) - 1) = n^2 - 3n + 4$.

Therefore the number of permutations with $a_1 \le 3$ or $a_2 \ge 3$ is $(n^2 - 3n + 4)((n-2)!) = 112(10!)$. Answer: $112(10!)$.

g) Take S to be the set of ordered pairs (b, c, d) of pairwise distinct ($b \neq c$, $c \neq d$, and $b \neq d$) integers in $\{1, \ldots, n\}$ such that $b = 2$ or $c = 3$ or $d = 4$. There are $(n-1)(n-2)$ elements of the form $(2, c, d)$. Similarly, there are $(n-1)(n-2)$ elements of the form $(b, 3, d)$, and $(n-1)(n-2)$ of the form $(b, c, 4)$. There are $(n-2)$ elements of the form $(2, 3, d)$ and same for $(2, c, 4)$ and $(b, 3, 4)$. Of course, there is only one element of S of the form $(2, 3, 4)$. By inclusion-exclusion principle there are $((n-1)(n-2) + (n-1)(n-2) + (n-1)(n-2)) - ((n-2) + (n-2) + (n-2)) + 1 = 3n^2 - 12n + 13$ elements in S.

Therefore the number of permutations with $a_1 = 2$, $a_2 = 3$, or $a_3 = 4$ is $(3n^2 - 12n + 13)((n-3)!) = 301(9!)$. Answer $301(9!)$.

II.

a) We need $k-2$ other elements from $\{2, ..., n-1\}$. There are $C(n-2, k-2)$ ways to choose them. Answer: $C(10, 2)$.

b) We need $k-1$ elements from $\{2, ..., n-1\}$. There are $C(n-2, k-1)$ ways to choose them. Answer: $C(10, 3)$.

c) By the inclusion-exclusion principle, $C(n-1, k-1) + C(n-1, k-1) - C(n-2, k-2)$. Answer: $2 \cdot C(11, 3) - C(10, 2)$.

d) There are $C(n-5, k)$ k-subsets which do not contain at least one integer ≤ 5. We subtract this from the total number of k-subsets. Answer: $C(12, 4) - C(7, 4)$.

e) The number of k-subsets not containing an integer ≤ 3 is $C(n-3,k)$. The number of k-subsets not missing any integer ≥ 10 is $C(n-(n-9),k-(n-9)) = C(9,k-(n-9))$. The number of k-subsets both not missing any integer ≥ 10 and not containing an integer ≤ 3 is $C(6,k-(n-9))$. Therefore the number of k-subsets containing an integer ≤ 3 and missing at least one integer ≥ 10 is $C(n,k) - C(n-3,k) - C(9,k-(n-9)) + C(6,k-(n-9))$. Answer: $C(12,4) - C(9,4) - 3$.

f) We subtract subsets with fewer than 2 numbers less than or equal to 6 from the total number of k-subsets. The number of k-subsets with exactly one number less than or equal to 6 are $6C(n-6,k-1)$ by the multiplication principle. The number of k-subsets with zero numbers less than or equal to 6 are $C(n-6,k)$. So $C(n,k) - 6C(n-6,k-1) - C(n-6,k)$ subsets have at least two numbers less than or equal to 6. Answer: $C(12,4) - 6C(6,3) - C(6,4)$

g) There are $\frac{n}{2}$ even integers in $\{1,\ldots,n\}$ (since n is even - otherwise we would round down) to choose from. Answer: $C(6,4)$.

III.

Because f maps a finite set to itself, injection, surjection, and bijection are all the same.

a) Assume $f(x) = f(y)$. Then $x+1 = y+1 \mod 12$ and $x = y \mod 12$. Therefore since $0 \leq x, y < 12$, $x = y$. Answer: Injection, surjection, bijection.

b) Assume $f(x) = f(y)$. Then $5x = 5y \mod 12$ and so 12 divides $5(x-y)$. Since 5 and 12 have no common factors, by the fundamental theorem of arithmetic, 12 must divide $x-y$. Thus $x = y \mod 12$. We conclude like in part a that $x = y$. Answer: Injection, surjection, bijection.

c) It is not injective because $0^2 = 6^2 \mod 12$. Answer: Neither.

d) It is not injective because $0^3 = 6^3 \mod 12$. Answer: Neither.

e) It is not surjective because since powers of 5 are always odd and thus not divisible by 12. That is there is no x for which $5^x = 0 \mod 12$ which is equivalent to $f(x) = 0$. Answer: Neither.

f) Observe that $f(2) = 3 = f(4)$. Answer: Neither

g) Observe that $f(3) = 0 = f(6)$. Answer: Neither.