Swan Song

Before:
- Random sampling from groups (Cayley graphs)
- Homogeneous spaces (Schreier graphs)

Idea: Use RW with group generators repeatedly add/modify gens if necessary (Babai, C-D, PRA)

Today:
- Random sampling from orbits of a particular group action

Examples:
1) k-subsets of \{1,..,n\} ∈ homog. space ↔ \text{Sym}(n) / \text{Sym}(k) × \text{Sym}(n-k), can use \text{Sym}(n) RW
2) \text{Gen}(k) ∈ generating k-tuples of \text{Sym}(n) hom. space ↔ \text{Aut}(\text{Gen}(k))/\text{Perm}(n), can use \text{Aut}(\text{Gen}(k)) RW
3) partitions \{x_1, x_2,.., x_{\ell}=n\} ∈ orbits of \text{Sym}(n)
More precisely, let \( S(n,k) \) be the set of unordered \( k \)-tuples of subsets \( (X_1, \ldots, X_k) \) such that:

\[
\forall i \in \mathbb{I}, X_i = \{1, \ldots, n\}, \quad X_i \cap X_j = \emptyset, \quad X_i \neq \emptyset
\]

**Example**

\[
|S(4,2)| = 7, \quad n=4, \quad k=2
\]

1, 2, 3, 4, \quad 1, 3, 4, 2, \quad 1, 2, 4, 3, \quad 1, 2, 3, 4, \quad 1, 3, 2, 4, \quad 1, 4, 2, 3

- \( |S(n,k)| \) is the Stirling number of the second kind.
- \( A \in S(n,k) \) can be sampled in poly-time.

**Observe**:

Partitions \( X_1, \ldots, X_k \) are orbits of \( S_n \) acting on \( S(n,k) \) / \( 1 + \cdots + k = n \).

**General Problem**

Count, approximately count, sample from the set of orbits.

**Example**

For \( kn \), \( k \geq 2 \), compute \# / sample in poly-time.
Burnside Processes

\[ G = \langle S \rangle \text{ finite group, } S = \{s_1, \ldots, s_k\} \]

\[ G \text{ acts on } X \vdash \forall x \exists g_i \in S x \text{ s.t. } \phi : G \rightarrow S x \text{ is homom. } \]

\[ \phi(s_i) = g_i \]

**Problem** \[ \#X^G := \# \text{orbits of } G \text{ acting on } X \vdash ?? \]

---

BP [Jerrum’93] \[ \text{step } x \rightarrow y \]

1) Choose unif. random \[ g \in St_G(x) := \{g \in G : g.x = x\} \]
2) \[ y \in Fix(g) := \{y \in X : g.y = y\} \]

Th/Obs [Jerrum]

BP is ergodic w/ stat dist \[ \frac{1}{\#X^G} \]

1) \[ x \rightarrow 1 \rightarrow y \] \[ \forall y \in X \text{ w/ some (small) prob.} \]
2) \[ \text{stat dist } \operatorname{deg}(x) = 15 \cdot |g(x)| = \frac{16}{16} |x| \text{ w/size of the orbit} \]

---
Exercise: Formulate BP on partitions. Prove that it works in poly-time.

**Warning:** Even when \( G = S_n \), step 1 can be hard.

Example: \( \alpha_n = \# \text{ non-isom. graphs} = \# \text{orbits of } S_n \text{ on all graphs on } n \text{ vert.} \)

Then \( \# St_{S_n}(G), G \in K_n \) is as hard as graph-isom.

L \( \in \exists? G \in St_{S_n}(G \cup G') \) which swaps \( G \leftrightarrow G' \)

Main example: Contingency Tables

\[
\begin{array}{c|ccc}
 & a_1 & \cdots & a_m \\
\hline
b_1 & \cdots & b_n \\
\end{array}
\]

\[\gamma(T(a, b)) = \# \{ (x_{ij})_n \in \mathbb{R}^{n \times n}, x_{ij} \geq 0 \text{ s.t.} \]
\[x_{ij} = a_i, \quad \sum_i x_{ij} = b_j \} \]

Q: Compute/Estimate/Sample from \( \gamma(T(a, b)) \)
Theorems
1) \(m, n = O(1)\) \(\# CT(a, b) = \text{Barvinok} '93\)
2) \(m = \Theta(n)\), \(q_{\text{max}}/q_{\text{min}}, b_{\text{max}}/b_{\text{min}} < 1.6\) [Barvinok et al '10]
3) \(m, n = O(1)\) [Chyam-Dyer '03]
4) \(a_{\text{min}}, b_{\text{min}} = \Omega(n^{2.520\ln n})\) [Pyer-Kannan-Mont '97]

\newtheorem{claim}{Claim}[section]
\begin{claim}
\end{claim}
Final Thoughts, Results, Conjectures

1) \( \text{PRA} \leq \text{rw on } \Gamma_k(G) \), \( \text{mix} = O(k^3 \log 161) \)

Q: lower bounds? \( \text{mix} = O(k \log k \log 161) \)

Can 3 be improved?

2) Q: Is there cutoff as \( |V| \to \infty \)?

Th [Peres-Tanaka-Zhai '20] yes, in TV, \( k \to \infty \) \& 6.

(formerly Diaconis - Saloff-Coste '96)

3) Th [Hubetzky-Sly '10, '11]

1) random 3-regular graphs have cutoff

2) Explicit construction of expanders w/ no cutoff and those w/ cutoff

Open problem: Do HGGG expanders have cutoff?
4) Th [Alan-Roichman] \(4\) \(G\) \(\exists S, \quad \langle S \rangle = 6\)
\(|S| = O(\log 161)\) s.t. \(1 - \epsilon > \frac{1}{10}\), \(\epsilon < \text{Cayley}(6,5)\)
Th [P, 1999] \(-1/-\) \(\text{mix Cayley}(6,5) = O(\log 161)\)
Th [Arvind, 2014] can be done in poly time.

5) Th [Radoi\c{c}i\c{c}-P.] \(\exists G \ni S, \quad \langle S \rangle = 6, \quad 151 \leq \log 161\)
\(s.t. \text{ Cayley}(6,5) \iff \text{Hamiltonian}\)
Th [Lu botzky '02] \(k = d(6) + O(\log \log 161) \Rightarrow \Phi_k(6)\geq 1-\varepsilon\)
Open \(S < G\) random, \(|S| = 1 - 1/11\) \(\Rightarrow\) Cayley (6,5)
is Hamiltonian w/ \(P > 1 - \varepsilon\)

6) Th \(\forall 6,\text{ solvable }, k \geq 3.75d(6) \Rightarrow \Phi_k(6) > \frac{1}{4}\)
[Mani, '96, Lucchini-Menegazzo-Kowgi '06]
Th [Crestani-Lucchini '15] \(\exists \text{6-solvable, which exhibits bias}\)
\(\text{Exe d}(6) = 2?\)
7) \( G = SL(k, 1F_p) \), \( R = \{ E_i;j \} \), \( k \) fixed

8) \( \exists \) explicit word proving \( \text{dist}(1, g) \leq c \log p \)

Th [Riley, 2005] yes, \( g \) \( k \geq 3 \)

Th [Larsen, 2003] almost, \( k = 2 \), \( \text{dist} = O(\log p \log \log p) \)

Th [Bramberg-Shpilrain-Vlăduţ] \( k \) or \( t + 01 \), \( k = 2 \)

\( G = SL(2, 1F_p) \), \( R = \{ (1, 1/2), (1/2, 1) \} \)

8) \( \text{conv} \) BP for CT w/ \( a_i; b_i = \text{poly}(n) \), \( m = \Theta(n) \)

mixes in \( \text{poly time} \). (expanding Diffie-Hellman)