L29
6/5/2020

Swan Song

Igor Pak

Before: Random sampling from
- groups (Cayley graphs)
- homogeneous spaces (Schreier graphs)

Idea: use RW w/ group generators repeatedly add/modify gen's if necessary (Babai, C-D, PRA)

Today: Random sampling from orbits of a particular group action

Examples:
1) k-subsets of $\{1..n\}$ ← homog. space
   $\leftrightarrow$ $S_n / S_k \times S_{n-k}$, can use S_n RW
2) $\Gamma_k(G)$ ← generating k-tuples of G ← hom. space
   $\leftrightarrow$ $\mathrm{Aut}(F_k) / (\text{law of } G)$, can use $\mathrm{Aut}(F_k)$ RW
3) partitions $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ ← orbits of S_n

More precisely, let $S(n,k)$ = set of <u>unordered</u> k-tuples of subsets $(X_1 \dots X_k)$

$\bigcup_{i=1}^{k} X_i = \{1 \dots n\}$, $X_i \cap X_j = \emptyset$, $X_i \neq \emptyset$

<u>Ex</u> $|S(4,2)| = 7$, $n=4, k=2$

123/4, 134/2, 124/3, 234/1, 12/34, 13/24, 14/23

<u>Exc</u> – $|S(n,k)|$ = Stirling #'s second kind

– $A \in S(n,k)$ can be sampled in poly time.

<u>Observe</u>: partitions $\lambda_1 \geq \dots \geq \lambda_k$ are orbits of S_n acting on $S(n,k)$ / $\lambda_1 + \dots + \lambda_k = n$ /

<u>General</u> <u>Problem</u>

count, approximately count, sample from the set of orbits

<u>Exc</u> For $k \sim n$, $\lambda_1 \geq \dots \geq \lambda_k$ compute # / sample in poly-time.

Burnside Processes

$G = \langle S \rangle$ finite group, $S = \{s_1 \dots s_k\}$

G acts on X $\leftarrow$ $\forall i \; \exists \, \sigma_i \in S_X$ $\quad i = 1..k$

s.t. $\varphi : G \to S_X$ is homom.

$\varphi(s_i) = \sigma_i$

Problem $\#X^G :=$ # orbits of G acting on X $\leftarrow$??

BP [Jerrum'93] step $X \to Y$

1) Choose unif. random $g \in St_G(x) := \{ g \in G : g \cdot x = x \}$

2) — // — $Y \in Fix(g) := \{ Y \in X : gY = Y \}$

Th / Obs [Jerrum]

BP is ergodic w/ stat dist $\propto \frac{1}{\# X^G}$

D 1) $X \to 1 \to Y$ $\forall Y \in X$ w/ some (small) prob.

2) stat dist ~~dowtdeg~~ $(x) = |St_G(x)| = |G| / |Gx|$ $\leftarrow$ size of the orbit

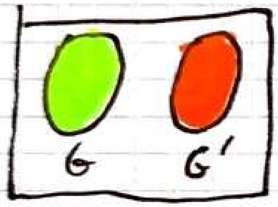
at X

Exc Formulate BP on partitions
Prove that it works in poly-time.

Warning: Even when $G = S_n$ step 1) can be **hard**

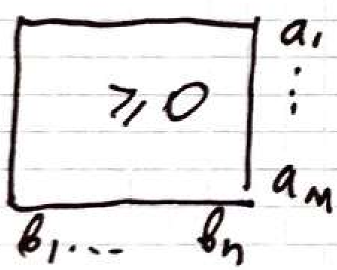
Ex $a_n :=$ # non-isom. graphs $=$ # orbits of S_n on all graphs on n vert.

Then $\# St_{S_n}(G)$, $G \subset K_n$ is as hard as graph-isom.



$\leftarrow \exists^{?}\ \sigma \in St_{S_{2n}}(G \sqcup G')$ which swaps $G \leftrightarrow G'$

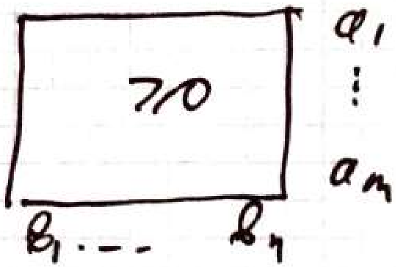
Main example: Contingency tables



$$CT(\bar{a}, \bar{b}) = \#\Big\{ (x_{ij}),\ x_{ij} \geqslant 0 \text{ s.t. } \sum_j x_{ij} = a_i,\ \sum_i x_{ij} = b_j \Big\}$$

Q: Compute / estimate / sample from $CT(\bar{a}, \bar{b})$

CT's are orbits



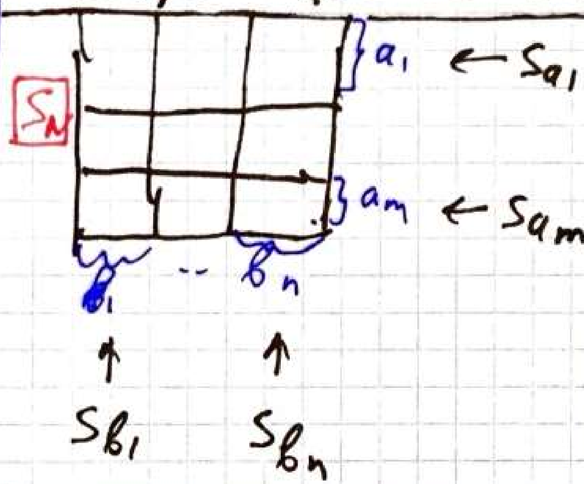
$n = a_1 + \dots + a_m$
$\quad = b_1 + \dots + b_n$

Diaconis – Gangolli MC

$\begin{pmatrix} \pm 1 & \mp 1 \\ \mp 1 & \pm 1 \end{pmatrix}$ if possible

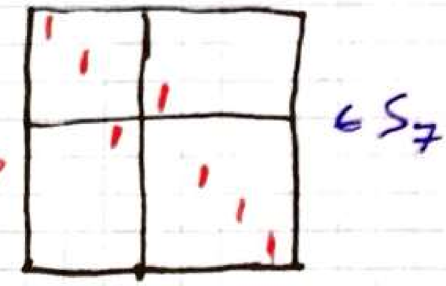
Let $\Sigma := S_{a_1} \times \dots \times S_{a_m} \times S_{b_1} \times \dots \times S_{b_n}$

Prop $\#CT(\bar{a}, \bar{b}) = \#$ orbits of Σ on S_n



Proof

$$\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 3 & 4 \\ \hline 3 & 4 & \end{array} \rightarrow$$



$\uparrow$
$CT(34, 34)$

Theorems

1) $m, n = O(1)$ $\quad \#CT(a, b) \leftarrow$ Barvinok '93
2) $m = \Theta(n)$, $\quad a_{max}/a_{min}$, $b_{max}/b_{min} < 1.6$ [Barvinok et al '10]
3) $\min\{m, n\} = O(1)$ [Cryan-Dyer '03]
4) $a_{min}, b_{min} = \Omega(n^{2.5} \log n)$ [Dyer-Kannan-Mount '97] [Morris '02]
   $m = n$

New thm [Dittmer – P. '19]

$a_i, b_j = O(n^{1/4 - \varepsilon})$
$m = \Theta(n)$

$\#CT$ can be approx in poly time using BP

Final Thoughts, Results, Conjectures

1) PRA $\leftarrow$ rw on $\Gamma_k(G)$, mix $= O(k^3 \log |G|)$

Q: lower bounds? mix $= \Omega(k \log k \log |G|)$

can 3 be improved?

2) Q: Is there cutoff as $|G| \to \infty$?

Th [Peres-Tanaka-Zhai '20] yes, in TV, $k \to \infty$ $\forall G$.

/formerly Diaconis – Saloff-Coste conj '96/

3) Th [Lubetzky-Sly '10, '11]

1) random 3-regular graphs have cutoff

2) $\exists$ explicit construction of expanders w/ no cutoff and those w/ cutoff

Open problem: Do MGG expanders have cutoff?

4) Th [Alon-Roichman, 1994] $\forall$ finite G $\exists/_R S$, $\langle S\rangle = G$
$|S| = O(\log|G|)$ s.t. $1-\lambda_1 > \frac{1}{10}$, $\lambda_1 \leftarrow \mathrm{Cayley}(G,S)$

Th [P, 1999] —//— $\mathrm{mix}\ \mathrm{Cayley}(G,S) = O(\log|G|)$

Th [Arvind, 2014] can be done in poly time.

---

5) Th [Radoičić-P.] $\forall G$ $\exists S$, $\langle S\rangle = G$, $|S| \le \log_2|G|$
s.t. $\mathrm{Cayley}(G,S) \leftarrow$ Hamiltonian.

Th [Lubotzky '02] $k = d(G) + O(\log\log|G|) \Rightarrow \varphi_k(G) > 1-\varepsilon$

Open $S \subset G$ random, $|S| =$ —//—//— $\Rightarrow \mathrm{Cayley}(G,S)$
is Hamiltonian w/ $P > 1-\varepsilon$

---

6) Th $\forall G$ - solvable, $k > 3.75\, d(G) \Rightarrow \varphi_k(G) > \frac{1}{4}$
[Mann '96, Lucchini-Menegazzo-Morigi '06]

Th $\exists G$ - ~~nilpotent~~ solvable which exhibits bias
[Crestani-Lucchini '15] / start w/ $G = S_4 \wr S_4 \wr S_4 \wr \ldots \wr S_4$ /

Exc $d(G) = ??$

7) $G = SL(k, \mathbb{F}_p)$, $R = \{E_{ij}\}$, k - fixed

Q: $\exists$ explicit word proving $\mathrm{dist}(1, g) \leq c \log p$

Th [Riley, 2005] yes, $k \geq 3$

Th [Larsen, 2003] almost, $k = 2$, $\mathrm{dist} = O(\log p \log\log p)$

Th [Bromberg-Shpilrain-Vdovina] $\frac{\text{sort of}}{\#17}$, $k = 2$

$G = SL(2, \mathbb{F}_p)$, $R = \left\{ \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix} \right\}$

---

8) Conj BP for CT w/ $a_i, b_i = \mathrm{poly}(n)$, $m = \Theta(n)$ mixes in poly time. / extending Dittmer P /