L27
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PRA - Final Chapter

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Last time: $\Gamma(G,S)$ has Kazhdan's property (T) w/
$\varepsilon = \varepsilon(G,S) > 0 \Rightarrow mix = O(\varepsilon^{-2} \log |\Gamma|)$
where $\Gamma \leftarrow$ Schreier graph, $S \leftarrow$ transitive

$\Gamma_k(H)$ is a Schreier graph of $Aut(F_k)$
$\Rightarrow$ need bounds on $\varepsilon(Aut(F_k), \text{Nielsen gen's})$

Today: - graph theory methods
- semi-definite programming

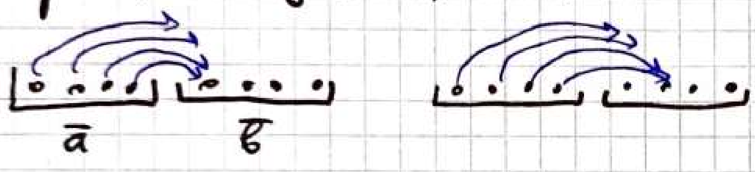
Th [Gamburd-P.] Bourgain-Gamburd]

[BG]: $H = SL(2,p)$, $H = \langle S \rangle$, $|S| \leq 4 \Rightarrow Cayley(H,S)$ - expander w $\varepsilon > 0$ universal

[GP]: $\Rightarrow \Gamma_k(H)$ - expander $\forall k \geq 8$ fixed

<u>Proof of [GP]</u> Fix $k=8$. Recall $m(H) \le 4$

Split $\Gamma_8(H)$ into union of products of Cayley graphs

$\Gamma_{\bar a}$ 

. . .

$\bar a = (h_1\, h_2\, h_3\, h_4)$

$\bar b = (h_5\, h_6\, h_7\, h_8)$

$\Gamma_{\bar a} = \mathrm{Cayley}(H, \bar a)^4$

 $\Gamma'_{\bar b}$

$\Gamma'_{\bar b} = \mathrm{Cayley}(H, \bar b)^4$

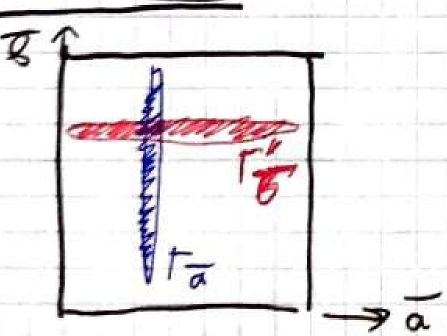
<u>L1</u> $\langle \bar a \rangle = H \Rightarrow \Gamma_{\bar a}$ - expander.

<u>L2</u> Γ, Γ' - expanders $\Rightarrow \Gamma \times \Gamma'$ - expander] $\underline{\underline{L2 \Rightarrow L1}}$

<u>L3</u> $\Gamma_8 = \bigsqcup_{\bar a} \Gamma_{\bar a} \quad \bigsqcup_{\bar b} \Gamma'_{\bar b}$, each $\Gamma_{\bar a}$ - expander

$(1-\varepsilon)$ fraction of $\Gamma'_{\bar b}$ - exp

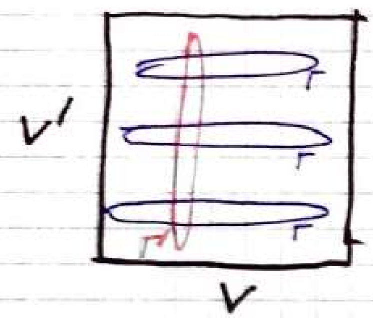
<u>Then</u> Γ_8 - expander



<u>Idea</u> prove <u>L2</u> which extends to proof of <u>L3</u>.

Proof of L2

$\Gamma = (V, E)$, $\Gamma' = (V', E')$, $|V| = n$, $|V'| = n'$, $\deg = d$



$$\hat{\Gamma} := \Gamma \times \Gamma' = (V \times V', \hat{E})$$

$A \subset V \times V'$, $|A| < |V| \cdot |V'| / 2 = \frac{nn'}{2}$

Consider $A(v') = A \cap \{(*, v'),\ v' \in V'\}$

$$\hat{E}(A, \bar{A}) \geq \sum_{v'} E(A(v'), \overline{A(v')})$$

incorrect $\longrightarrow$ $$\geq \sum_{v'} d\,|A(v')|\,\phi(\Gamma) = d\,|A|\,\phi(\Gamma)$$

Recall $$\phi(\Gamma) = \min_{\substack{A \subset V,\ A \neq \emptyset \\ |A| \leq n/2}} \frac{E(A, \bar{A})}{d \cdot |A|}$$

Sometimes $|A(v')| > \frac{n}{2}$ (!)

Real argument: two types of $A(v')$

① $|A(v')| > \frac{3}{4} n$

② $|A(v')| \leq \frac{3}{4} n$

one of the types has $> \frac{|A|}{2}$

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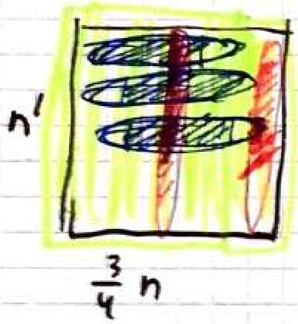
case ② $\hat{E} \geq \frac{1}{2} \boxed{d\,\phi(\Gamma)} \frac{|A|}{2} \geq \frac{\phi(\Gamma)}{8} (2d)\,|A| \Rightarrow \phi(\hat{\Gamma}) \geq \frac{\phi(\Gamma)}{8}$

Case 1



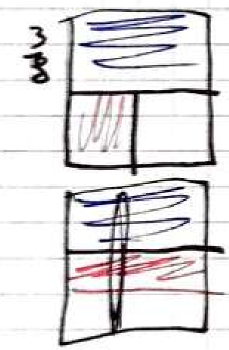
Consider now

$$A'(v) = A \cap \{(v, x)\} \quad , v \in V$$

$\# A(v')$ w/ $\geq \left(\frac{3}{4} n\right)$ elts take up $> \frac{|A|}{2}$ elts

we have $< \frac{|A|}{2} < \frac{n n'}{4}$ elts remain

In summary



$A'(v)$ have $\leq \frac{3}{4}|A|/n \leq \frac{3}{8} n'$

⊕ $\frac{1}{4} n n'$ distributed outside of $[A(v')$ w/ $\geq \frac{3}{4} n]$

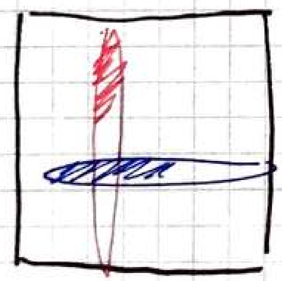
$\left(\frac{1}{4} n' n\right) / (5/8\, n') = \frac{2}{5} n$ best case

$\left(\frac{3}{8} + \frac{1}{4}\right) n' = \frac{5}{8} n'$ worst case

$\Rightarrow$ same as case ① w/ $\frac{3}{4} \leftarrow \frac{5}{8}$ $\square$

For ≤ 3

Same idea



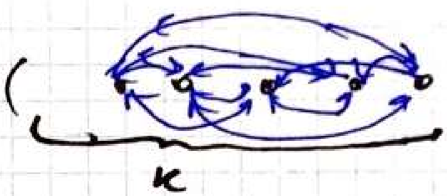
Key obs: ① never used row/column isa

② proof robust to ignore ε-fractions of columns. $\Rightarrow$ $\square$

Th [Kaluba-Nowak-Ozawa '17]

$\mathrm{Aut}(F_5)$ has (T) w/ $\varepsilon \geq 0.18$

Cor all $\Gamma_k(H)$, $k \geq 5$ are expanders

Proof idea $\Gamma_k = \bigsqcup_{5\text{-subsets } B}$ ()

Now apply $\underbrace{\text{similar argument}}_{\text{graph theoretic}}$ + symmetrization.

$$\Rightarrow \phi(\Gamma_k(H)) \geq \Omega\left(\frac{1}{k^2}\right)$$

$$\Rightarrow \mathrm{mix}\, \Gamma_k(H) = O(k^5 \log H) \quad \square$$

Th [Kaluba-Kielak-Nowak '18]

$G = \mathrm{Aut}(F_k)$ has (T) w/ $\varepsilon(G, R) \geq \sqrt{\dfrac{1.378\,(k-2)}{6\,(k^2-k)}}$

$$= \Omega\left(\frac{1}{\sqrt{k}}\right)$$

$$\Rightarrow \mathrm{mix}\, \Gamma_k(H) = O(k^3 \log|H|) \quad \square$$

Proof idea g (T) $\leftarrow$ SDP argument by ~~Ozawa~~ Ozawa '16

Th [Netzer–Thom '15]

$G = SL(3, \mathbb{Z})$ has (T) w/ $\varepsilon = \varepsilon(G, R) > \frac{1}{6}$

Cor mix $T(SL(3,p), R) = O(\log p)$ and $O(1)$
can be computed now: $1 - \lambda_1 \geq \frac{1}{72}$ (via P-Z)
improving $1/15{,}000 \leftarrow$ Burger + PZ

Idea $G = \langle R \rangle$

Consider $\mathbb{R}[G] \leftarrow$ $*$-algebra $(\sum c_g\, g)^* = \sum c_g\, g^{-1}$

R = usual transvections $\{E_{ij}^{\pm}\}$, $|R| = 12$

$$\Delta := 12 - \left[\sum_{M \in R} M\right] = \frac{1}{2} \sum_{M \in R} (1 - M)^*(1 - M)$$

Let $A_\varepsilon := \Delta^2 - \varepsilon \Delta$

L1 $\varepsilon = \frac{1}{6} \Rightarrow A_\varepsilon = \sum_{i=1}^{n} B_i^* B_i$
for $n = 121$

L2 $\simeq \Rightarrow =$

$\blacksquare$ 6

Making S_N into expanders

Q [Lubotzky, 1980s] $\exists ?\ \langle R_N \rangle = S_N$ s.t. $|R_N| = O(1)$
Cayley (S_N, R_N) are expanders

Non-example $R = \{(12), (1 \ldots n)^{\pm 1}\}$, $\langle R \rangle = S_n$
$\Rightarrow (1-\lambda_1) = \Theta(\frac{1}{n^2})$

Positive indication: $\exists R$ s.t. diam $= O(n \log n)$
/ we proved long ago /

Prop [Lubotzky-P.'01] $Aut(F_k)$ has (T) some k
$\Rightarrow \{S_N\}$ can be made into expanders
w/ $|R_N| \leq 4k(k-1)$

Proof $H \leftarrow$ simple group, e.g. $PSL(2,p)$, $A_n, n \geq 5 \ldots$
$G := H^m$, where $m = |H|^k \varphi_k(H)$ / as in Hall thm /
$\Gamma := \Gamma_k(H^m)$. Gilman thm says $Aut(F_k)$ acts as S_m
/ $k = 5$ we have $4 \cdot 5 \cdot 4 = 80$ generators /

Prop + Th [Kaluba +] $\Rightarrow$ S_n can be made into expanders

Th [Kassabov, '07]

$N = (2^{3k}-1)^6$, S_N —"— w/ $|R| < 156$

Proof uses: $\begin{cases} S_n\text{ - character estimates} \\ \text{Kazhdan const for } SL(3k, \mathbb{F}_2) \\ \text{random walk estimates.} \end{cases}$

vs. $\begin{cases} \text{Gilman's proof is algebraic} \\ \text{+ Th [Kaluba +] computational.} \end{cases}$