L26
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Bounds on Kazhdan Const

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Last time:

Def $\begin{cases} G = \langle S \rangle & \text{has } (T) \text{ w/ } \varepsilon = \varepsilon(G,S) > 0 \\ \text{if} & \forall\, \pi : G \to U(V) \text{ unitary rep}^*,\ V - \text{Hilb. space} \\ & \forall\, v \in V,\ \|v\| = 1 \\ & \exists\, s \in S \text{ s.t. } \|\pi(s)v - v\| > \varepsilon \end{cases}$

$*$ ← no invariant vectors

Th [Kazhdan, Margulis, etc.] (G, R) have (T)

1) $G = SL(k, \mathbb{Z})$, $R = \left\{ E_{ij}^{\pm} = \begin{pmatrix} 1 & & \pm1 \\ & \ddots & \\ & & 1 \end{pmatrix},\ 1 \le i \ne j \le k \right\}$

2) $G = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{pmatrix} \simeq SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, $R = \left\{ \begin{pmatrix} 1 & \pm1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \pm1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & \pm1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm1 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

Cor $G = SL(k, \mathbb{F}_p)$, $R = $ —//—

mix $= O_k(\log p)$

Q How does const depend on k?

Bounded generation

Th [Carter-Keller, 1983] $\quad k \geq 3$

$G = SL(k, \mathbb{Z})$, $R = \{ E_{ij} \}$

$\forall g \in G \quad \exists \quad g = E_{i_1 j_1}^{a_1} \; E_{i_2 j_2}^{a_2} \cdots \; E_{i_\ell j_\ell}^{a_\ell}$

where $a_i \in \mathbb{Z}$, $\ell \leq \frac{3n^2 - n}{2} + 36$

Exc: False for $SL(2, \mathbb{Z})$

Proof $\Leftarrow$ elementary modulo Dirichlet thm ($\forall$ arithm progr. $\exists$ primes ...)

Th $G = SL(k, \mathbb{Z}/p\mathbb{Z})$, $R = \{ E_{ij} \}$
— // — $\ell = O(k^2)$

} all we want for now

$\square$ $\; G = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ * & & 1 \end{pmatrix} \begin{pmatrix} \sigma \in S_k \end{pmatrix} \begin{pmatrix} 1 & & * \\ & 1 & \\ 0 & & 1 \end{pmatrix}$

Recall ① diam $U(k, \mathbb{Z}/p\mathbb{Z})$ wrt R $= O(k^2)$ ✓

② diam S_k in $\{(ij)\} = O(k)$ ✓

③ $\left(\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline 0 & & \ddots \; 1 \end{array}\right)$ = const dist in R $\quad \forall k \geq 3$

Exc
Hint: check $k=3$ $\blacksquare$

Th [Shalom, 1999]

$G = SL(k,\mathbb{Z})$, $R = \{E_{ij}\}$ has (T)

w/ $\varepsilon = \varepsilon(k) > c/k^2$ for some $c > 0$

Proof idea

LO $\pi : G \to U(V)$ unitary, $V \leftarrow$ Hilb. space

suppose $\|\pi(g)v - v\| < 1$ $\forall g \in G$

Then $\exists\, w \in V$, $w \neq 0$ invariant vector: $/\pi(g)w = w$ $\forall g \in G/$

D Let $Q := \{\pi(g)v,\ g \in G\} \subset V$

Then $Q \leftarrow$ bounded, $0 \notin Q$, $Q \leftarrow G$-invariant

$w :=$ unique pt in V s.t. $f_Q(w) = \sup_{q \in Q}\{\|w - q\|\}$ is minimized.

$\Rightarrow w \neq 0$

$\Rightarrow w$ is desired invariant vector ∎

Proof of Shalom Th (idea, cont'd)

Obs1 / L1 Every E_{ij}^{a} $\in$ some $SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$ / $SL(2,\mathbb{F}_p) \ltimes \mathbb{F}_p^2$ /

D (for p) E_{ij}^{a} are all conjugate $\Leftarrow$ $\in$ one conjugacy class / $\blacksquare$

Obs2 / L2 if $\forall \|E_{ij}^{a} v - v\| < \varepsilon \Rightarrow \|gv - v\| < O(k^2)\varepsilon$

D $g = \underbrace{E_{i_1 j_1}^{a_1}}_{=s_1} \underbrace{E_{i_2 j_2}^{a_2}}_{=s_2} \cdots \underbrace{E_{i_\ell j_\ell}^{a_\ell}}_{=s_\ell}$ | $\|gv - v\| = \|s_1 s_2 \cdots s_\ell v - v\| =$

By C-K Thm

$\leq \sum_{i=1}^{\ell} \|s_1 \ldots s_i v - s_1 \ldots s_{i-1} v\| < O(k^2)\varepsilon$ $\blacksquare$

$L0 + L1 + L2 \Rightarrow \exists\, E_{ij}$ s.t. $\|E_{ij} v - v\| > \frac{c}{k^2}$ $\blacksquare$

Last time: $(1-\lambda_1) \geq \dfrac{\varepsilon(G,R)^2}{2|R|} \Rightarrow 1-\lambda_1 \geq \dfrac{c}{2(k^2)^2 k^2} = \Omega(k^{-6})$

$\Rightarrow \operatorname{mix}(SL(k,\mathbb{F}_p), R) = O(k^6 \log|SL(k,\mathbb{F}_p)|)$

$= O(k^8 \log p)$

Th [P - Żuk, 2002]

$G = \langle S \rangle$ has (T) w/ Kazhdan const $\varepsilon(G,S) > 0$

Suppose S ← transitive ($\forall s, s' \in S$ $\exists h \in Aut(G)$ s.t $s = h(s')$)

Then $(1 - \lambda_1) > \frac{\varepsilon(G,S)^2}{2}$

D (proof idea)

$\pi : G \to U(V)$ unitary rep

Let $\bar{\pi} : G \to U(V^N)$, $N = \#$ elts in $\Phi \subset Aut(G)$ preserving S

$\pi(g)v \to (\pi \circ \phi_1(g)v, \ldots \pi \circ \phi_N(g)v)$ $\Phi = \{\phi_1, \ldots \phi_N\}$

This reduces usual $(1-\lambda_1) > \frac{\varepsilon(G,S)^2}{2|S|}$ to [>] as desired ∎

Cor $mix(SL(k, \mathbb{F}_p), R) = O(k^6 \log p)$

in place of k^8

Th [Kassabov, 2005]

$G = SL(k,\mathbb{Z})$, $R = \{E_{ij}^{\pm}\}$ has (T) w/ $\varepsilon(G,R) > \frac{1}{64\sqrt{k}+2856}$
$= \Omega\left(\frac{1}{\sqrt{k}}\right)$

Proof idea: combination of Shalom + G-G proof

in place of $\{E_{ij}^{a}\}$ ← (△°), (S_k), ⊕ careful Bookoping

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Cor $\operatorname{mix}(G, R) = O\left(\frac{\varepsilon(G,R)^{-2}}{2} \log|G|\right) = O(k^3 \log p)$

$G = SL(k, \mathbb{F}_p)$ /current Best/

Exc $\varepsilon(G,R) = O\left(\frac{1}{\sqrt{k}}\right)$ so Th is tight up to a const

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Back TO PRA

Cor (from P-Z) $G = \mathrm{Aut}(F_k)$ has (T) w/ $\varepsilon(G,\{R,L\}) = \varepsilon(k)$

$\Rightarrow \operatorname{mix} \Gamma_k(H) = O\left(\frac{1}{\varepsilon(k)^2} \log|H^k|\right) = O\left(\frac{k \log|H|}{\varepsilon(k)^2}\right)$

Combinatorial approach

Conj / Near-Th [Bourgain-Gamburd, 2008]

$G = SL(2, \mathbb{F}_p)$, $G = \langle S \rangle$, $|S| \le 4$

Then diam Cayley$(G,S) = O(\log p)$

Moreover — " — $\leftarrow$ expanders w/ ε universal const

Recall $m(G) \le 4$

$\leftarrow$ proved for const fraction of primes

Th [Gamburd-P, 2006]

Conj $\Rightarrow$ $\Gamma_k(H)$ are expanders, $k \geq 8$, $k = O(1)$