

L25
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Expanders & PRA

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Last time: $\Gamma = (V, E) \leftarrow d$ -regular graph, $|V|=n$
$$\phi(\Gamma) := \min_{\substack{X \subset V \\ 1 \leq |X| \leq \frac{n}{2}}} \frac{E(X, \bar{X})}{d|X|}$$
 / edge expansion
conductance

$\Gamma \leftarrow \varepsilon$ -expander if $\phi(\Gamma) > \varepsilon$, $\varepsilon > 0$ fixed
 d -fixed, $n \rightarrow \infty$

Th [Kaluza et al. '17, '18]

$\forall k \geq 5 \exists \varepsilon = \varepsilon(k)$ s.t. $\Gamma_k(G)$ are ε -expanders

/ here $|G| \rightarrow \infty$, $\varepsilon(k)$ not known, $d = 4k(k-1)$ /

since $\Gamma_k(G)$ can be disconn, $\forall$ conn. comp -1-

Today: I will explain what's going on

Cheeger inequality

$\Gamma = (V, E)$ $\leftarrow$ d -regular graph (conn)

$$M_\Gamma := (m_{vw}), \quad m_{vw} = \begin{cases} \deg(v) & , v=w \\ -1 & , (v,w) \in E \\ 0 & , \text{oth.} \end{cases} \quad \text{[Laplacian]}$$

$1 = \lambda_0 > \lambda_1 > \lambda_2 > \dots > -1$ eigenvalues of $\frac{1}{d} M_\Gamma$
 $\uparrow$
P-FTH

$$\text{Th } \frac{1 - \lambda_1}{2} \leq \phi(\Gamma) \leq \sqrt{2(1 - \lambda_1)}$$

/ Alon, Alon-Milman, Jerrum-Sinclair, Lawler, Cheeger, ... /

$$\text{Cor } \text{mix} = O(\log n \cdot \underbrace{\left(\frac{1}{1 - \lambda_1}\right)}_{\text{relaxation time}}) = \boxed{O\left(\frac{1}{\phi(\Gamma)^2} \log n\right)}$$

compare with P-SC Bound $\text{mix} = \boxed{O(\Delta^2 |S| \log |G|)}$

Proof of Cheeger inequality (sketch)

① $\frac{1-\lambda_1}{2} \leq \phi(\Gamma)$

Idea: take bad X
 $\Rightarrow$ construct small $(1-\lambda_1)$
eig.

② $\phi(\Gamma) < \sqrt{2(1-\lambda_1)}$

Idea: take bad eig. f
 $\Rightarrow$ construct bad X

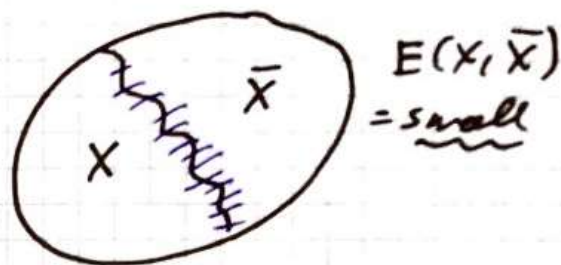
Moral: $\lambda_1(\Gamma) < 1-\epsilon$
 and $\phi(\Gamma) > \epsilon$

are equivalent def's
 of expanders

Take $f \in \mathbb{R}[\Gamma]$

$$f(v) = \frac{|x|}{n}, v \in X$$

$$f(v) = -\frac{n-|x|}{n}, v \notin X$$



Note $\langle f, 1 \rangle = 0$

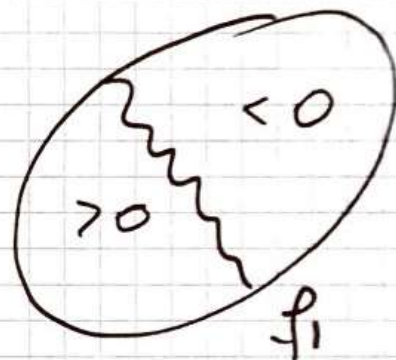
compute $\langle \frac{1}{d} M_\Gamma \cdot f, f \rangle = 1 - \text{small}$

use max principle $1 - \text{small} < \lambda_1$

Take $f_1 \in \mathbb{R}[\Gamma]$

corr to eig. λ_1

Let $X := \left\{ v \in V \mid f_1(v) > 0 \right\}$



Estimate $E[X, \bar{X}]$

using CS ineq.

Exc finish these proofs!!

Last time: explicit construction (Margulis-Gabber-Gol'd)

CS implications

$L \leftarrow \text{log-space}$

$SL \leftarrow \text{sym log-space}$ (undirected s-t conn $\neq$ complete)

$NL \leftarrow \text{nondet sym log-space}$ (directed $—||—$)

Easy: $L \stackrel{\textcircled{1}}{\subseteq} SL \stackrel{\textcircled{2}}{\subseteq} NL$

$\textcircled{1}$ make graph out of state space

$\textcircled{2}$ convt time = poly(n)

Conj $NL \neq L$

Th [Reingold '04] $L = SL$

$\iff$ one can create universal traversing sequence
of length poly(n)

Proof idea: $\textcircled{1}$ turn any $\Gamma = (V, E)$ into expander by powering

$\textcircled{2}$ find UTS for expanders

• [Hoory-Kidgerson '93]



• $\textcircled{4}$

Back to M-G-G graphs

Last time: we sketched why they are expanders
after G-G. What did Margulis do?

Def [Kazhdan's Property (T)]

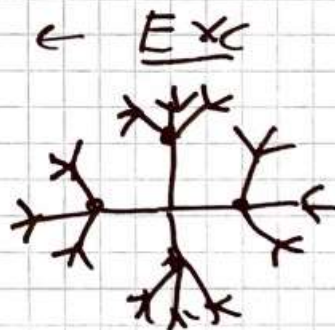
$G = \langle S \rangle \leftarrow$ fin. or inf. group

G has (T) if $\forall \pi: G \rightarrow U(V)$ unitary rep, $V \in$ Hilbert space
 $\forall v \in V, \|v\|=1$
 $\forall g \in S, \|\pi(g)v - v\| > \varepsilon$
w/ $\varepsilon = \varepsilon(S)$

Ex ① G - finite $(1 - \lambda_1) \geq \frac{\varepsilon(S)}{2|S|} \leftarrow$ Exc

② $G = F_2$ does NOT have (T)

Hint: Take $V = L^2(F_2)$



More examples of (T) and NOT-(T)

③ Th [Kazhdan '67] $\forall k \geq 3$

$SL(k, \mathbb{Z})$ has (T)

← NOT Exc
/uses structural prop./

④ Cor [Margulis '73] $\forall k \geq 3$ $R = \{E_{ij}; i \neq j\} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$

$SL(k, \mathbb{Z}/p\mathbb{Z}) = SL(k, \mathbb{F}_p)$ has (T)

with same $\varepsilon(k)$ ← indep of p !

⇒ Cayley & Schreier graphs of $SL(3, p)$
are ε -expanders

Exc show that construction of MGG is ε -exp.

⑤ $SL(2, \mathbb{Z})$ does NOT have (T) ← Exc

Hint: $SL(2, \mathbb{Z})$ contains F_2 of finite index

Th $SL(2, p)$ w/ $R = \left\{ \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \right\}$ are expanders

• /hard result• based on Langlands• Th /

• ⑥

⑥ $\text{Aut}(F_k) \leftarrow$ group of Aut of F_k

Th [Nielsen, 1924]

$\text{Aut}(F_k)$ is generated by

$$\begin{cases} R_{ij}^{\pm} : (x_1 \dots x_i \dots x_k) \rightarrow (x_1 \dots x_i x_j^{\pm 1} \dots x_k) \\ L_{ij}^{\pm} : (x_1 \dots x_i \dots x_k) \rightarrow (x_1 \dots x_j^{\pm 1} x_i \dots x_k) \end{cases}$$

classical
result
in Combin
Group Theory

Old Problem: $\text{Aut}(F_k)$ has (T)?

$k=2$ - NO, maps onto $SL(2, \mathbb{Z}) \in \text{Exc}$

$k=3$ - NO, maps onto F_2 [Lubotzky
Grunewald]

$k=4$ - Open

$k=5$ - 2017 [Kaluba et al]

$k \geq 6$ - 2018 [—]

Proof of $k \geq 5$ are computer assisted

Back to PRA

Observation [Lubotzky-P. '01]

$\text{Aut}(F_k)$ has (T) for $k \Rightarrow \Gamma_k(G)$ are exp.

u/ some $\varepsilon = \varepsilon(k) \Rightarrow \text{mix} \approx O(\varepsilon^{-2} \log |G| \cdot k)$

Th [L-P] $\text{Aut}(\text{Free nilp on } k\text{-gen of class } l)$
has (T)

$\Rightarrow \forall G\text{-nilp, class } l$

$$\text{mix } \Gamma_k(G) = O(\varepsilon^{-2}(k, l) \cdot \log |G| \cdot k)$$

Obs follows from viewing $\Gamma_k(G)$ as
Schreier graphs of $\text{Aut}(F_k)$

Question: how $\varepsilon = \varepsilon(k)??$

Next time