L24 Expander graphs

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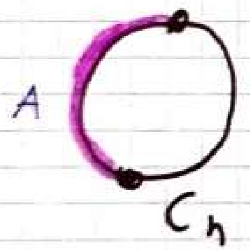
Def $G=(V,E)$ ← conn. graph, d-regular, $\varepsilon>0$

G is ε-expander if $\forall\, A\subset V$, $A\neq\emptyset$

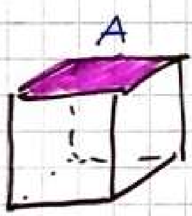
$\#E(A, V-A) \geq \varepsilon\,|A|$, $|A|\leq \frac{|V|}{2}$

$n=|V|$, $\{G_n\}$ ← expander if $d=\text{const}$, $\exists\,\varepsilon>0$ s.t. G_n is ε-expander $\forall n$.

Ex 1) n-cycle C_n is $\left(\frac{4}{n}\right)$-expander ← Exc



2) d-hypercube is 1-expander ← Exc



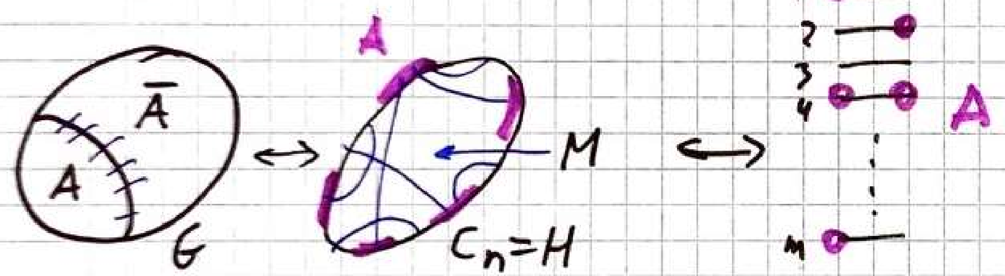
3) random 3-regular graph is $>\frac{1}{100}$-expander w.h.p. ← Exc*

← uniform

← sum of 3 random matchings

Th $G=(V,E) \leftarrow$ random 3-regular graph, $|V| = n = 2m$
formed by $G = H \cup M$, H = n-cycle, $M \leftarrow$ uniform PM
then G is ε-expander whp $\varepsilon =$

D (sketch)



$|A| = k$ $\quad 1 \le k \le m$

$$\mathbb{E}[\#\bullet\!\!-\!\!-] = 2\,\frac{k(n-k)}{n(n-1)}\, m \geq 2\,\frac{k}{n}\,\frac{n}{2} = k$$

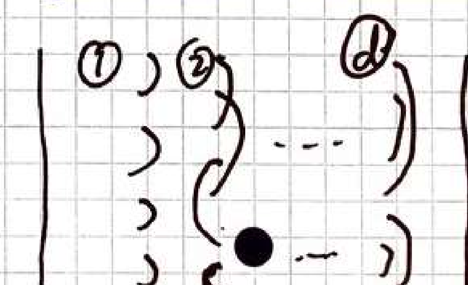
$$P\Big[\, E(A,\bar{A}) \geq \frac{k}{2} \,\Big] > 1 - n^{-ck}$$
$\forall A$

$$\Rightarrow \quad P\Big[\, E(A,\bar{A}) \geq \frac{|A|}{2} \ \ \underset{1 \le |A| \le n/2}{\forall A \subset V} \Big] \geq 1 - \sum_{k=1}^{m} \binom{n}{k} n^{-ck} > 1 - \delta > 0$$

$\square$

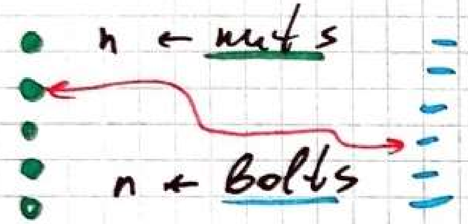
Motivation: [AKS] $\leftarrow$ Ajtai-Komlós-Szemerédi (1983)

Sorting network $\quad$ $O(n \log)$ sorting w/ depth $O(\log n)$



deterministic algorithm.

Nuts & Bolts [Alon-Blum-Fiat-Kannon-Naor-Ostrovsky] 1994



Random Alg

1) pick random bolt, compare w/ all nuts
2) compare matching nut w/ all bolst

split the problem into 2 roughly $\frac{n}{2}$ size

cost $O(n \log n) \leftarrow f(n) = 2n + 2f(\frac{n}{2}) =$
$= 2n + 2n + 4f(\frac{n}{2}) = \dots$

Th [Alon ++]

$\exists\ O(n (\log n)^{3+\varepsilon})$ deterministic Alg

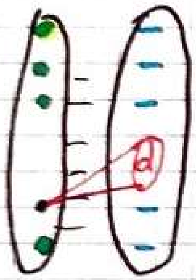
[AKS], [Alon++] required

Th [Margulis, 1973]

$\exists$ explicit construction of ε-expanders, d, ε-fixed
$n \to \infty$

Def $G = (V, E)$ has an explicit construction if $V \subseteq \{0,1\}^{\ell}$, $\ell = O(\log n)$
$\forall v_i \in V, (v_i\ v_j) \overset{?}{\in} E$ in $Poly(\log n)$ time

Nuts & Bolts det Alg



1) compare all bolts w/ corr. nuts
2) delete half nuts whose rank far from $\frac{d}{2}$

Repeat until middle nut is found.
Split the problem.

Exc get $O(n(\log^4 n))$ alg.

Margulis - Gabber-Galil expanders $G_p = (V_p, E_p)$

$V_p = \mathbb{Z}_p^2$

$d = 8$

$E_p = \{ (a,b) \leftrightarrow (a\pm 1, b), (a, b\pm 1), (a, b\pm a), (a\pm b, b) \}$

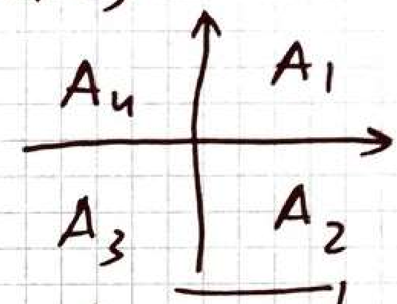
Th [M, G-G'81]

$\{G_p = (V_p, E_p)\} \leftarrow \frac{1}{100}$-expanders.

Margulis'73 $\leftarrow$ Algebraic proof using Kazhdan (T)

Gabber-Galil $\leftarrow$ first "elementary proof"

D $S(a,b) := (a, a+b)$ $\qquad T(a,b) := (a+b, b)$



$A_i := A \cap Q_i$
$- (0, \mathbb{Z})$
$- (\mathbb{Z}, 0)$

$A_0 := -A \cap -$[illegible]$-$

L $\forall\ A \subset \mathbb{Z}^2 \setminus O = (0,0)$

$$|E(A, \bar{A})| \geq \frac{1}{7}|A|$$

$\boxed{DL}$ $|S(A_1)| = |T(A_1)| = |A|$ since $S, T \leftarrow$ perm's

$S(A_1) \cap T(A_1) = \emptyset \quad \leftarrow \quad (a, a+b) = (a'+b', b') \Rightarrow b = -b'$ ×

$S(A_1), T(A_1) \subset Q_1 \quad \Rightarrow$ at least $\frac{1}{2}$ of $E(A_1, *) \ni A$

$\Rightarrow$ Claim 1 $\qquad \#E[A - A_0, \bar{A}] \geq |A - A_0| = |A| - |A_0|$

Claim 2 $\qquad \#E[A_0, \bar{A}] \geq 4|A_0| - 3|A - A_0| = 7|A_0| - 3|A|$

Exc

Claim 1 + Claim 2 $\Rightarrow$ L (By balancing ineq's) $\square$

Now $\quad \mathbb{Z}^2 \to \mathbb{Z}_p$ is difficult but routine $\square$

Th (Margulis '73) $\forall k$

$\mathrm{Cayley}(G_p, R_p) \leftarrow$ expander

$G_p = SL(k, \mathbb{F}_p)$, $R_p = \left\{ \begin{pmatrix} 1 & \pm 1 & \\ & 1 & \\ & & \ddots \end{pmatrix}_{k \times k} \right\}$

G-G $\Leftarrow$ Margulis $k=3$

$\left(SL(3,p) \right) \begin{pmatrix} a \\ b \\ c \end{pmatrix} \leftarrow \exp$, + proj onto (a,b), $\underline{c=1}$

Cor $\Gamma_k(\mathbb{Z}_p^k)$ are expanders $\forall k$

$\Rightarrow$ mix time of PRA on $\Gamma_k(\mathbb{Z}_p^k)$
$= c_k (\log p)$

vs $O(k^6 (\log p)^3)$ lost time.

Th [P-Zuk + Kassabov]

$\mathrm{mix} = O(k^3 \log p)$

Next week:

Cor/Th [Kaluba-Kielak-Nowak-Ozawa '17 + '18]

$\forall k$ PRA mixes on $\Gamma_k(G)$ $\forall k \geq 5$
in $O_k(\log|G|)$ time.

Real Th $\Gamma_k(G)$ are ε_k-expanders $\forall k \geq 5$

/ Conjecture Lubotzky-P '01 /

Note: 1) every conn. comp. of $\Gamma_k(G)$

2) proof $\leftarrow$ computer-assisted