L23 PRA Mixes in Poly Time

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Recall: $G = \langle S \rangle$ fin group, $S = \{s_1, \ldots s_k\}$

$\Gamma_k(G)$ - graph on gen'g k-tuples $(g_1, \ldots g_k)$

$k > d(G) \underset{conj}{\Rightarrow} \Gamma_k(G)$ conn.

PRA: Run RW on $\Gamma_k(G)$ starting w/ $(s_1, \ldots s_k)$

mix time of this RW? / on conn. comp of $\Gamma_k(G) \ni S$ /

Main Th: $k = \Omega(\log|G| \log\log|G|)$

[P. FOCS'00] then mix $= \text{Poly}(k, \log|G|)$

/ $k = \Theta(\log|G| \log\log|G|)$

$\Rightarrow$ mix $= O(\log^6|G| (\log\log|G|)^5)$

Embedding REO into PRA

Babai Alg

$$(s_1 \ldots s_k \; 1 \, 1 \ldots 1)$$
$$\downarrow$$
$$(s_1 \ldots s_k \; * \; 1 \ldots 1)$$
$$\downarrow$$
$$\ldots$$
$$(s_1 \ldots s_k \; \underbrace{* \; * \ldots *}_{O(\log|G|)})$$

Note: works even if we start w/ $(s_1 \ldots s_k \; \underbrace{\circ \; \circ \ldots \circ}_{\text{any}})$

Cooperman–Dixon Alg

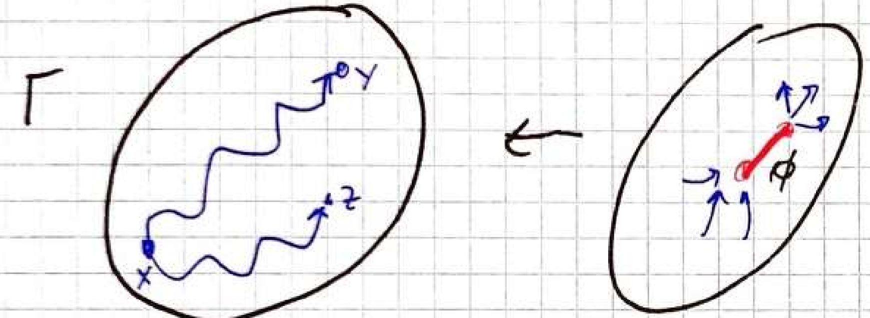
We want: Collection of paths $\Upsilon = \{ \gamma : \bar{g} \to \bar{g}' \;\; \bar{g}, \bar{g}' \in \Gamma_k(G) \}$

s.t. $\Delta = \max|\gamma| = \mathrm{poly}(k, \log|G|)$

congestion through every edge $\phi = \mathrm{poly}(k, \log|G|)$

$\Rightarrow \mathrm{Mix} = O(\Delta^2 \phi \binom{k}{2} \underbrace{\log|\Gamma_k(G)|}_{O(k \log|G|)})$.

Multicommodity flow explanation



$\Gamma = (V, E)$ ← regular graph
degree d / ← $\binom{k}{2} 8$ /

$\gamma = \{ [\gamma_{xy} : x \to y, \forall x y \in V, w_{xy}] \}$

s.t. $\sum_{\gamma} w_{xy}(\gamma) = 1 \quad \forall x, y \in V$

$$\phi = \min_{e \in E} \Big[\sum_{x,y} \sum_{\gamma_{xy} \ni e} w_{xy}(\gamma) \Big]$$ ← conjestion through e

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Th mix = poly$(d, \phi, \log|V|)$

/ ① $O(\phi^2 d \log|V|)$, diam already in ϕ /

② $O(\Delta \phi d \log|V|)$, if extra symmetries enforse even load /

Proof ← exactly the same Dirichlet form argument

Now $\boxed{\text{No}}$ averaging over G $\boxed{\text{Yes}}$ weighted average over γ

Babai Alg $\Rightarrow$ multicomm. flow

$k > c \log_2 |G|$, c - fixed, large enough.

$\Rightarrow$ every $\bar{g} \in \Gamma_k(G)$ is redundant

$\bar{g} = (\bullet \; \bullet \; s_1 \circ s_2 \circ \circ \ldots \circ s_\ell \bullet \bullet \ldots \bullet)$ $\qquad \ell = \log_2 |G|$

$\langle s_1 \ldots s_\ell \rangle = G.$

$\bar{g}' = (\bullet \; s_1' \; s_2' \bullet \circ \ldots \ldots \circ \circ \bullet \; s_\ell' \; \bullet)$

Run Babai Alg $\quad \bar{g} \to (* \; * \; s_1 * s_2 * \ldots \; *) = \bar{h}$

$\bar{g} \to (* \; s_1' \; * \; * \qquad) = \bar{h}'$

$\sigma \in S_k \qquad \bar{h} \to \sigma(\bar{h})$

If $\bar{g} \to \bar{h}' \stackrel{!}{=} \sigma(\bar{h}) \Rightarrow \gamma : \bar{g} \to \bar{h} \to \sigma(\bar{h}) = \bar{h}' \to \bar{g}'$ ✓

otherwise No path.

Lemma 1) $\Delta = poly(k, \log|G|)$ $\leftarrow$ Babai th $\quad O(\log^3 |G| \, k)$

2) $\phi = $ —"— 1/—— —"— ④ Symmetry prop.

④ $\varphi_k(G) = 1 - \frac{\cdot\cdot c'}{(\log|G|)^c}$

Congestion

Step t $(s_1 \ldots s_\ell \; * \ldots * \; \circ \; \bullet \ldots \bullet) = \bar{a}$ $\quad (g_i \leftarrow g_i\, g_j^{\pm 1}) \leftarrow R_{ij}^{t}$

Recall only combin. seq of path.

$$\left[R_{i_1 j_1}^{\varepsilon_1} \; R_{i_2 j_2}^{\varepsilon_2} \; \ldots \underset{\substack{\uparrow \\ t}}{R_{i_t j_t}^{\varepsilon_t}} \; R_{i_m j_m}^{\varepsilon_m} \right]$$

m = mix time / length of Babai Alg.

$$\bar{g} \xrightarrow[t-1]{} \bar{a} \xrightarrow[R_{i_t j_t}^{\varepsilon_t}]{} \bar{a}' \xrightarrow[m-t]{} \bar{g}'$$ $\leftarrow$ part of some Babai Alg path whp

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whp $\leftarrow \left(1 - \frac{1}{(\log|G|)^c}\right)$

$\Rightarrow$ in mix time of Babai $\varepsilon = (\log|G|)^{-c}$

$\Rightarrow$ $(\log\log|G|)$ terms in k.

$\underline{L} \Rightarrow \underline{Th}$ $\quad \blacksquare$ $\boxed{\text{Done}}$

Back to Basics

$G = \mathbb{Z}_p^m$, p - prime. $\qquad d(G) = m$.

$\Gamma_m(G) \longleftrightarrow GL(m, \mathbb{F}_p)$ $\qquad$ det $\leftarrow$ const, $\Gamma' \leftarrow$ conn. comp w/ det $= 1$

PRA $\longleftrightarrow$ RW on $SL(m, \mathbb{F}_p)$ $\quad$ w/ $R = \left\{ E_{ij} = \begin{array}{c} \begin{pmatrix} 1 & & \pm 1 \\ & \ddots & \\ & & 1 \end{pmatrix} i \\ j \end{array} \right\}$

L $\quad$ diam $\Gamma' = O((m^2 p)^2)$

D $\quad SL = \bigcup_{\sigma \in S_m} \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} (\sigma) \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\blacksquare$

L' $\quad$ diam $\Gamma' = O(m^2 (\log p))$

Exc

Th $\quad$ mix $\Gamma' = \mathrm{poly}(m, \log p)$

D $\quad$ mix $= O(\mathrm{diam}^2 \cdot \log|\Gamma'|)$

$= O(m^4 (\log p)^2 (m^2 \log p))$

$= O(m^6 (\log p)^3)$ $\square$

Better Bounds??

Next time

$O(m^9 \log p)$

<u>Th</u> (D, S-C ← ∃ Inventiones)
~1998

$$\operatorname{mix} \Gamma_k(G) = \operatorname{poly}(k, \log G, D)$$

D = max diam of all cayley graphs $\Gamma(G, S)$
$|S| = k$