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Product Replacement graphs

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Def $G = \langle S \rangle$, $d = d(G)$, $k \geq d(G)$

$\Gamma_k(G) = (V, E)$, $V = \{(g_1 \ldots g_k) : \langle g_1 \ldots g_k \rangle = G\}$

$E = \left\{ (g_1 \ldots g_i \ldots g_k) \in V \leftrightarrow (g_1 \ldots g_i' \ldots g_k) \in V \quad g_i' = g_i g_j^{\pm 1}, \ i \neq j, \ g_i' = g_j^{\pm 1} g_i \right\}$

Conj $k \geq d(G) + 1 \Rightarrow \Gamma_k(G)$ conn

Th $k \geq d(G) + m(G) \Rightarrow$ — " — , $m(G)$ = max non-red. gen. set.

Th $G = A_n$, S_n, $SL(2,p)$, etc, $k \geq 3$

$\Rightarrow \Gamma_3(G)$ has a large conn. comp.

Th [Babai, '96] $k \geq 2 \log_2 |G|$

Then $\operatorname{diam} \Gamma_k(G) = O(\log^2 |G|)$

Proof 0) $(g_1 \dots g_k) \xrightarrow[O(k)]{} (g_1' \dots g_\ell' \; 1 \dots 1)$

$\ell = \frac{1}{2} k$
$= \Theta \log_2 |G|$

1) $(s_1 s_2 \dots s_\ell \; 1 \dots 1) \to (s_1 s_2 \dots s_\ell \; h_1 1 \dots 1)$

$\to (s_1 s_2 \dots s_\ell \; h_1 h_2 1 \dots 1) \to \dots$

$\to (s_1 s_2 \dots s_\ell \; h_1 h_2 \dots h_\ell)$ $\boxed{\ell \cdot O(\ell)}$

$h_i \in$ random subproduct

$h_i = g_1^{\varepsilon_1} s_2^{\varepsilon_2} \dots s_k^{\varepsilon_k} \times h_1^{\alpha_1} \dots h_{i-1}^{\alpha_{i-1}}$

2) $\to (1 \; 1 \dots 1 \; h_1 h_2 \dots h_\ell)$ $\boxed{O(\ell^2)}$

s.t. $C(\bar{h}) = \{ h_1^{\varepsilon_1} \dots h_\ell^{\varepsilon_\ell} \} = G$

$s_i = h_1^{\varepsilon_1} \dots h_\ell^{\varepsilon_\ell}$

3) $(1 1 \dots 1 \; h_1 h_2 \; h_\ell) \to (h_1 h_2 \dots h_\ell \; 1 \dots 1) \to (h_1 \dots h_\ell \; h_1' \dots h_\ell')$

$\to (1 \dots 1 \; h_1' \dots h_\ell')$ $\boxed{O(\ell^2)}$

In summary: $\bar{g} \xrightarrow[1)]{} \bar{s} \xrightarrow[2)]{} \bar{h} \xrightarrow[3)]{} \bar{h}' \xleftrightarrow[2)']{} \bar{s}' \xleftarrow[1)']{} \bar{g}'$

Total: $O(\ell^2)$

Analysis? $\longrightarrow$

$\boxed{\text{I}}$ Assuming Cooperman-Dixon Thm

$K \geq C \cdot \ell$, $\ell = \log_2 |G|$ / where $C = 10$ /

$\Rightarrow$ $\underbrace{h_{m+1} \cdots h_{m+2\ell}}_{\bar{h}}$ $\leftarrow \approx$ unif in G $m = k - 2\ell$

$\Rightarrow$ $\exists R + h$ $C(\bar{h}) = \{ h_{m+1}^{\varepsilon_1} \cdots h_{m+2\ell}^{\varepsilon_\ell} \} = G$ $\Rightarrow \blacksquare$

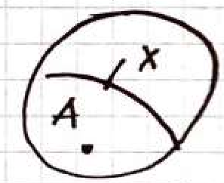
$\boxed{\text{II}}$ Babai way (w/ Out C-D thm)

$(s_1 \ldots s_\ell \; h_1 \ldots h_{i-1} 1 \ldots 1) \to (s_1 s_2 \ldots s_\ell \; h_1 \ldots h_{i-1} \; \underline{\underline{h_i}} \; 1 \ldots 1)$?

We choose $h_1 \ldots h_i$ s.t. $C(h_1 \ldots h_i) = 2^i$

Let $A = C \cdot C^{-1} \subset G$. If $A \neq G$
"$C(h_1 \ldots h_{i-1})$"
$\exists \; x \in \partial_S A$

Take $h_i \leftarrow x$



Cay(G, S)

$\Rightarrow$ $C(h_1 \ldots h_i) = C(h_1 \ldots h_{i-1}) \cup C(h_1 \ldots h_{i-1}) h_i$

Same argument : $\exists \; h_1 \ldots h_\ell$, $A = G$. $\blacksquare$

Bias in the PRA

Th [Hall]

H ← non-ab simple group

$\alpha_k(H) = \max\{m : d(H^m) = k\}$

Then $\alpha_k(H) = \#$ orbits of $Aut(H)$ action on $\Gamma_k(G)$

Cor $\alpha_2(A_n) > \frac{n!}{8}$ for n large enough

/ th $\forall n \geq 5$ /

Ex (Hall) $\alpha_2(A_5) = 19 > \frac{5!}{8} = 15$

$\triangleright$ (of Cor) $Aut(A_n) = S_n$ $\forall n = 5, \geq 7$

Exc Find $Aut(A_6) \neq S_6$

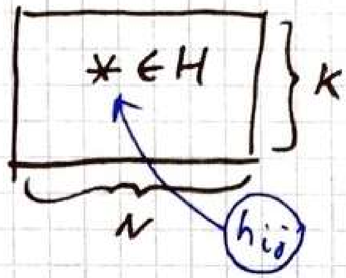
$|\Gamma_2(A_n)| > \frac{1}{2} |A_n|^2 = \frac{n!^2}{8}$

$\Rightarrow \#$ orbits $> \frac{n!}{8}$ $\blacksquare$

Proof of Hall thm (abridged)

$G = H^N$

$(g_1 \cdots g_k) \in G^k$



$g_i = (h_{i1}, \dots h_{iN})$

Note: $\forall j \;\; \langle h_{1j} \dots h_{kj} \rangle = H$

L (Hall)

$(h_{ij}) \in \Gamma_k(G) \iff$ all $(h_{1j} \dots h_{kj})$ lie in diff. orbits of $Aut(H)$

$\Rightarrow$ obvious since oth. $O_j \overset{\omega}{\simeq} O_{j'}$ $\quad \omega \in Aut(H)$

$\Leftarrow$ Exc

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