

L2 (4/1/2020)

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Probability of generation

G - finite group

$d(G) = \min \# \text{ gen's}$

$\varphi_k(G) := P(\langle g_1, \dots, g_k \rangle = G)$, $k \geq d(G)$

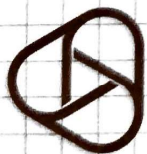
$\mathfrak{d}(G) := \min \{ k : \varphi_k(G) \geq \frac{1}{2} \}$

Problem given G , find $\mathfrak{d}(G)$
/estimate, math/

Main Th (Dixon, 1968, conj by Netto, 1892)

$$\varphi_2(S_n) \rightarrow \frac{3}{4} \quad \text{as } n \rightarrow \infty$$

$$\varphi_2(A_n) \rightarrow 1 \quad \text{--- 11 ---}$$



Preliminaries

$$(1) \quad G = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \dots \times \mathbb{Z}_p, \quad d(G) = 1$$

$$\varphi_1(G) = (1 - \frac{1}{2})(1 - \frac{1}{3}) \dots (1 - \frac{1}{p}) \rightarrow 0 \quad \text{as } p \rightarrow \infty$$

$$\varphi_1(G) = \exp \left[\sum_{\substack{q < p \\ q \text{ prime}}} \log(1 - \frac{1}{q}) \right]$$

up to a const

$$\approx \exp \left[- \sum_{\substack{q < p \\ \text{prime}}} \frac{1}{q} \right] \rightarrow 0$$



$$(2) \quad \varphi_2(G) = (1 - \frac{1}{2^2})(1 - \frac{1}{3^2}) \dots (1 - \frac{1}{p^2})$$

$$> \prod_{n=2}^{\infty} (1 - \frac{1}{n^2}) > 0$$

const



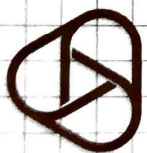
$$(3) \quad G = \mathbb{Z}_2^m, \quad d(G) = m$$

$$\varphi_m(G) = \left(1 - \frac{1}{2^m}\right) \left(1 - \frac{1}{2^{m-1}}\right) \cdots \left(1 - \frac{1}{2}\right)$$

$$> \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right) = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{32} + \frac{1}{128} - \dots$$

const $\rightarrow$

Exc $\varphi_{m+1}(G) > \frac{1}{2} \Rightarrow d(G) = m+1$



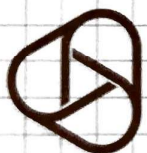
Random graph process (on n vert)

$$G_0 = \emptyset, \quad G_{i+1} = G_i + e, \quad e \in \underset{i'n}{\text{random edge}} \underset{K_n}{}$$

$\tau \leftarrow$ stopping time, until

- G - connected
- G has a Δ
- G has a component of size n^ϵ
- G has a H.G.
- ⋮

Erdős - Rényi resolved a lot
+++ of these questions



Random group process (in G)

$$H_0 = 1, \quad H_{i+1} = \langle H_i, g \rangle, \quad g \in G \text{ unif.}$$

$\tau :=$ first time i , $H_i = G$

$$\varphi_k(G) = P(\tau \leq k) \iff P(\tau > k) = 1 - \varphi_k(G)$$

Prop 1 $\varphi(G) \leq 2 E[\tau]$

$$\text{D } P(\tau > 2 E[\tau]) < \frac{1}{2} \quad \square$$

Prop 2 $E[\tau] \leq 2 \log_2 |G|$

$$\text{D } t < \tau \Rightarrow P(H_{t+1} \neq H_t) = 1 - \frac{|H_t|}{|G|} \geq \frac{1}{2}$$

Thus expected time to
at least double H_t is ≤ 2

$$\Rightarrow E[\tau] \leq 2 \log_2 |G| \quad \square$$



For $G = S_n$, $\log |S_n| = n \log n + O(n)$

so $\kappa(S_n) = O(n \log n)$

$$\text{D } \kappa \leq 2 E[\tau] \leq 4 \log_2 |S_n| \quad \square$$

Prop 3 $E[\tau] \leq 2 \ell(G)$, where

$$\ell(G) = \max \text{ length of } 1 = H_0 \not\subseteq H_1 \not\subseteq H_2 \dots G$$

Lemma $\ell(S_n) = O(n \log \log n)$.

$$\Rightarrow \kappa(S_n) = O(n \log \log n)$$

$$\text{D } O_p := \max \{k : p^k \mid n!\}$$

Then $O_p = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \dots$

$$\leq \frac{n}{p(1 - \frac{1}{p})} = \frac{n}{p-1}$$



Then

$$\begin{aligned} \ell(S_n) &\leq \sum_{\substack{p \leq n \\ p\text{-prime}}} O_p \leq n \sum_{\substack{p \leq n \\ p\text{-prime}}} \frac{1}{p-1} \\ &\leq n \log \log n + O(n) \end{aligned}$$

Indeed

$$\sum_{\substack{p \leq n \\ p\text{-prime}}} \frac{1}{p-1} \sim \int_1^n \frac{dx}{x \log x}$$

$$= \int_1^n \frac{d \log x}{\log x} = \log \log x \Big|_1^n$$



Remark $\ell(S_n) \leq \frac{3}{2}n$

Th [Cameron-Solomon-Turull, 1989]

$$\ell(S_n) = \lfloor \frac{3n-1}{2} \rfloor - b_n, \text{ where}$$

$b_n := \# \text{ of 1s in binary exp. of } n$

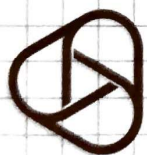
/ uses CFSG /

Def $m(G) := \max \text{ size of non-redundant}$
gen set S , $\langle S \rangle = G$

Then $d(G) \leq m(G) \leq \ell(G) \leq \log_2 |G|$

Th (Whiston, 2000)

$m(S_n) = n-1$. / uses CFSG /
// $S_n = \langle (12), (23), \dots, (n-1, n) \rangle$ //



Exc $m(SL(2, P)) = O(1)$

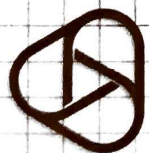
Exc $E[\tau] \leq \ell(G) + C, \quad C \leftarrow \text{univ. const}$

Exc $|G| \leq 2^m$ Then

$\psi_k(G) \geq \psi_k(\mathbb{Z}_2^m) \quad \forall k \geq m$

Exc $\chi(G) \geq E[\tau]/2$

(a companion to $\chi \leq 2E[\tau]$)



What's coming

Th [Dixon]

$$\psi_2(A_n) = 1 - O\left(\frac{1}{\log \log n}\right) \quad \checkmark$$

Th [Buboi, 1989]

$$\psi_2(A_n) = 1 - \frac{1}{n} - O\left(\frac{1}{n^2}\right)$$

/uses CFSG/

