

L17
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Dirichlet forms

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G -finite group

$V = \mathbb{R}[G] \leftarrow$ real-val. functions on G .

Inner product on V

$$\langle \phi, \psi \rangle := \sum_{x \in G} \phi(x) \psi(x) = |G| \mathbb{E}[\phi(X) \psi(X)]$$

$X \in G$ uniform

$\pi \leftarrow$ prob. distr. on G

$P_\pi : x \rightarrow xy, y \leftarrow \pi$, so 1 step of $\{X_t\}$, $X_{t+1} = X_t \pi$

P_π acts on V :

$$[P_\pi \psi](x) = \sum_{y \in G} \psi(xy) \pi(y) = \mathbb{E}[\psi(X_1) | X_0 = x]$$

$$\text{supp}(\varphi) := \{x \in G : \varphi(x) \neq 0\}, \quad \varphi \in V$$

Def (Dirichlet form)

$$\mathcal{E}_\pi(\varphi, \varphi) := \langle (1 - P_\pi) \varphi, \varphi \rangle = |G| \cdot \mathbb{E}[(\varphi(X_0) - \varphi(X_1)) \varphi(X_0)]$$

where $X_0 \leftarrow \text{Unif}$, $X_1 = X_0 \pi$

Note: $\mathbb{E}[(z_0 - z_1) z_0] = \frac{1}{2} \mathbb{E}[(z_0 - z_1)^2]$
 $\forall z_0, z_1$ w/ some distribution

$$\begin{aligned} \Rightarrow \mathcal{E}_\pi(\varphi, \varphi) &= \frac{|G|}{2} \mathbb{E}[(\varphi(X_0) - \varphi(X_1))^2] \\ &= \frac{1}{2} \sum_{x, y \in G} (\varphi(x) - \varphi(xy))^2 \pi(y) \end{aligned}$$

$$G = \langle S \rangle, \quad S = S^{-1}, \quad S = \{s_1, \dots, s_k\}$$

$$\Gamma = \Gamma(G, S) \leftarrow \text{Cayley graph}$$

$\gamma_y \leftarrow$ path in Γ from 1 to $y \in G$

$$y = s_{i_1} s_{i_2} \dots s_{i_\ell} \iff \gamma_y: 1 \rightarrow s_{i_1} \rightarrow s_{i_1} s_{i_2} \rightarrow \dots \rightarrow y$$

$$\Upsilon = \{\gamma_y\} \leftarrow \text{collection of paths}$$

$$C \subset G \text{ subset}, \quad s \in S$$

$$N_\Upsilon(s, C) := \max_{y \in C} \mu_s(\gamma_y)$$

where $\mu_s(z) := \# \text{ times generator } s \in S \text{ used in } \gamma_z: 1 \rightarrow z$

Ex $G = \mathbb{Z}_m, \quad S = \{\pm 1\}, \quad C = G$

$\Upsilon = \text{shortest paths} \Rightarrow N_\Upsilon(s, C) \leq \lfloor \frac{m}{2} \rfloor$



Th [Diaconis, Saloff-Coste '93]

(400+ citations)

$$C \subset G, \quad \bar{C} = C \cup \partial_r C, \quad d = \text{diam}(\bar{C})$$

$\pi, \tilde{\pi} \leftarrow \text{sym. probab. distributions}$

$$\pi(s^{-1}) = \pi(s) \\ \forall s \in S$$

$$S \subset \text{supp}(\pi)$$

Then

$$\sum_{\pi} (\varphi, \varphi) \geq \frac{1}{A} \sum_{\tilde{\pi}} (\varphi, \varphi)$$

where

$$A = d \max_{s \in S} \frac{N_r(s, \bar{C})}{\pi(s)}$$

Proof

$$Y = s_1 s_2 \dots s_e, \quad s_i \in S$$

$$\varphi(x) - \varphi(xy) = [\varphi(x) - \varphi(xs_1)] + [\varphi(xs_1) - \varphi(xs_1 s_2)] \\ + \dots + [\varphi(xs_1 \dots s_{e-1}) - \varphi(xy)]$$

$$\stackrel{CS}{\Rightarrow} [\varphi(x) - \varphi(xy)]^2 \leq e^* \sum_{i=1}^e [\varphi(xs_1 \dots s_{i-1}) - \varphi(xs_1 \dots s_e)]^2$$

$$\bullet \quad e^* \leq \text{diam}(\bar{C}) =: d \quad \bullet \quad \bullet \quad \textcircled{4}$$

$$\sum_{x \in G} [\varphi(x) - \varphi(xy)]^2 \leq d \sum_{z \in G, s \in S} N_T(s, \bar{c}) [\varphi(z) - \varphi(zs)]^2$$

Average LHS w.r.t. (y) w/ weights $\hat{\pi}(y)$

$$\begin{aligned} \sum_{x, y \in G} [\varphi(x) - \varphi(xy)]^2 \hat{\pi}(y) &\leq d \sum_{z \in G, s \in S} \frac{1}{|S|} \\ &\leq A \cdot \sum_{z \in G, s \in S} [\varphi(z) - \varphi(zs)]^2 \pi(s) \end{aligned}$$

□

Note $S = S^{-1}$ was never used.

Cor [to be proved next lecture]

$$\mathbb{E}_{\hat{\pi}}(\varphi, \varphi) \leq A \mathbb{E}_{\pi}(\varphi, \varphi) \quad \forall \varphi \in V$$

Then $(1 - \lambda_i) \leq A (1 - \lambda_i)$

□

Applications

to mix time

$$\text{mix } \Gamma(G, S) = ?$$

, $G = \text{fin. group}$, $G = \langle S \rangle$, $S = S^{-1}$

$$Q^t(g) := P(X_t = g), \quad g \in G$$

$$\text{mix} := \min \{ t \mid \text{sep}(Q^t, U) \leq \frac{1}{2} \}$$

$$\text{Prop} \quad \text{mix} = O\left(\frac{1}{1-\lambda_1} \log |G|\right)$$

$$\text{Take } \tilde{\pi} = U \text{ on } G, \quad \tilde{\pi}(g) = \frac{1}{|G|} \quad \forall g \in G$$

$$\text{trans. matrix } \frac{1}{|G|} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}, \text{ so } \tilde{\pi}_0 = 1, \tilde{\pi}_1 = \tilde{\pi}_2 = \dots = 0$$

$$\text{Cor} \Rightarrow \lambda_1 \leq 1 - \frac{1}{A}$$

$$A = d \max_{s \in S} \max_{g \in G} \frac{\mu_s(g)}{\pi(s)}$$

$$\boxed{\frac{1}{1-\lambda_1} \leq A}$$

$$d = \text{diam } \Gamma(G, S)$$

$$\frac{1}{\pi(s)} = |S|$$

$\Rightarrow$

$$A = d \cdot |S| \cdot \max_{s \in S} N_\Gamma(s, G)$$

$$\Rightarrow \frac{1}{1-\lambda_1} \leq D^2 |S|$$

⑥

Ex 1 $G = \mathbb{Z}_m$, $S = \{\pm 1\}$, $N \approx \frac{m}{2}$, $A = \Theta(m^2)$

$$\Rightarrow \frac{1}{1-\lambda_1} \leq 1 - \frac{1}{\Theta(m^2)} \Leftrightarrow \frac{1}{1-\lambda_1} = \Theta(m^2)$$

$$\Rightarrow \text{mix} = O(m^2 \log m)$$

Ex 2 $G = \mathbb{Z}_2^m$, $S = \{(0, \dots, 1, \dots, 0)\}$, $N = 1$, $d = m$
 $A = |S| d = O(m^2)$

$$\Rightarrow \frac{1}{1-\lambda_1} = O(m^2)$$

$$\Rightarrow \text{mix} = O(m^2 \log |G|) = O(m^3) \quad \text{vs. } O(m \log m)$$

Ex 3 $G = S_n$, $R = \{(1,2), (2,3), \dots, (n-1,n)\}$
 $|R| = n-1$, $d = \binom{n}{2} = \Theta(n^2)$, $N = \Theta(n^2)$

$(n, n-1, \dots, 2, 1)$
 $\downarrow$
 $\Rightarrow \frac{1}{1-\lambda_1} = O(n^4) \Rightarrow \text{mix} = O(n^4 \log |S_n|)$
 $= O(n^5 \log n)$

Ex $G = \left(\binom{[n]}{2}, E \right) \quad K_2$ $R = \left\{ \binom{[n]}{2} \mid \exists i, j \right\}$

Th $\text{mix} = O(n^2 \log n)$

$\text{diam} = O(n^2) \quad V = O(n)$

$|R| = \binom{n}{2} = \Theta(n^2) \quad A = O(n^5)$

$\text{mix} = O(n^5 \log |G|) = O(n^5 \cdot n^2)$
 $= O(n^7)$

$$G = St(2, p) \quad |G| = O(p^3)$$

$$R = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\pm 1}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\pm 1} \right\}$$

$$\text{diam } \Gamma(G, R) = O(\log p)$$

$$A = \underbrace{(\log p)}_d \underbrace{(\log p)}_n \cdot \underbrace{|R|}_4 = O(\log p)^3$$

$$\text{mix} = O((\log p)^3)$$

$$O(\log p)$$