Old Problem:

Find *nice* bijections between combinatorial objects.
Specifically, between 200+ counted by the *Catalan numbers*.

New Problem:

Explain why some objects have *super nice* (canonical) bijections
while others do not (and what this all even means).

\[
C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{4^n}{\sqrt{\pi n^3}} \left( 1 - \frac{9}{8n} + \frac{145}{128n^2} - \cdots \right)
\]
Plan:

1. Classical Catalan structures
2. Selected known results
3. Pattern avoidance
4. The results
5. Connections to probability
6. Applications
7. Alternating and Baxter permutations
1. Classical Catalan structures:

1) \( C_n = \) number of triangulations of \((n + 2)\)-gon (Euler, 1756)
2) \( C_n = \) number of non-associative products of \((n + 1)\) numbers (Catalan, 1836)

\[
((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd))
\]

3) \( C_n = \) number of binary trees on \((2n + 1)\) vertices
4) \( C_n = \) number of *plane trees* with \( (n + 1) \) vertices
5) $C_n =$ number of *Dyck paths* of length $2n$

i.e. lattice paths $(0, 0) \rightarrow (n, n)$ below $y = x$ line.
Canonical bijections:

- Triangulations $\leftrightarrow$ Binary trees
- Binary trees $\leftrightarrow$ Non-associative products
- Binary trees $\leftrightarrow$ Plane trees
- Plane trees $\leftrightarrow$ Dyck paths

These can be extremely useful for studying asymptotics of combinatorial statistics and more generally the shape of combinatorial objects.
2. Selected asymptotic results:

**Theorem** (Aldous, 1991; DFHNS, 1999)
The p.d.f. of the maximal chord-length in a random triangulation of regular $n$-gon converges to

\[ \frac{3x - 1}{\pi x^2 (1 - x)^2 \sqrt{1 - 2x}}, \quad \frac{1}{3} < x < \frac{1}{2}, \quad \text{as } n \to \infty. \]

**Theorem** (DFHNS, 1999)
$\Delta_n =$ maximal degree of a random triangulation of $n$-gon. Then for all $c > 0$

\[ P(|\Delta_n - \log_2 n| < c \log \log n \to 1 \quad \text{as } n \to \infty. \]

DFHNS = Devroye, Flajolet, Hurtado, Noy and Steiger.
**Theorem:** Let $\delta_n$ be the degree of a root in a random plane tree with $n$ vertices.

$$P(\delta_n = r) \to \frac{r}{2^{r+1}}, \quad E[r] \to 3 \quad \text{as} \quad n \to \infty.$$ 

**Theorem:** Let $h_n$ height of a random plane tree with $n$ vertices, $m_n$ the height of a random Dyck path of length $2n$. Then:

$$h_n, m_n \sim \sqrt{\frac{\pi n}{2}}$$

3. Pattern avoidance:

Permutation $\sigma \in S_n$ contains pattern $\omega \in S_n$ if matrix $M(\sigma)$ contains $M(\omega)$ as a submatrix. Otherwise, $\sigma$ avoids $\omega$.

Example

$\sigma = (2,4,5,1,3,6)$ contains 132 but not 321.

$$M(\sigma) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

contains

$$\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}$$

but not

$$\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}$$
Patterns of length 3

\( s_n(\omega) := \text{number of permutations } \sigma \in S_n \text{ avoiding } \omega \)

**Theorem** (MacMahon, 1915; Knuth, 1968)
\( s_n(\omega) = C_n \) for all \( \omega \in S_3 \).

**Two Observations:**
\( s_n(123) = s(321), \ s_n(132) = s(231) = s_n(312) = s(213) \) via symmetries

[Kitaev]: Nine different bijections between 123- and 132-avoiding permutations.

**Question:** Can it be true that all nine and nice? How about canonical?

**My Answer:** No canonical bijection is possible. Here is why...
Simulations by Madras and Pehlivan

Monte Carlo simulation 1

Monte Carlo simulation 2

Figure: Randomly generated 312 avoiding permutation with N=100

Figure: Randomly generated 312 avoiding permutation with N=200
4. Shape of random pattern avoiding permutations

\[ P_n(i, j) := \frac{1}{C_n} \sum_{\sigma} M(\sigma)_{ij}, \]

where the sum is over all 123-avoiding permutations.

\[ Q_n(i, j) := \frac{1}{C_n} \sum_{\sigma} M(\sigma)_{ij}, \]

where the sum is over all 132-avoiding permutations.

Main Question: What do \( P_n(\ast, \ast) \) and \( Q_n(\ast, \ast) \) look like, as \( n \to \infty \)?
Shape of random 123-avoiding permutations (surface)

Surface $P_{250}(i,j)$ and the same surface in greater detail.
Shape of random 132-avoiding permutations (surface)

Surface $Q_{250}(i, j)$ and the same surface in greater detail.
Diagonal of $P_n(\ast, \ast)$ in details
Main Theorem for $P_n(\ast, \ast)$, [Miner-P.]

$$P_n(an, bn) < \varepsilon^n, \quad a + b \neq 1, \quad \varepsilon = \varepsilon(a, b), \quad 0 < \varepsilon < 1$$

$$P_n(an - cn^\alpha, (1 - a)n - cn^\alpha) < \varepsilon^{n^{2\alpha - 1}}, \quad \frac{1}{2} < \alpha < 1, \quad \varepsilon = \varepsilon(a, b, \alpha), \quad 0 < \varepsilon < 1$$

$$P_n(an - cn^\alpha, (1 - a)n - cn^\alpha) \sim \eta(a, c) \kappa(a, c) \frac{1}{\sqrt{n}}, \quad \alpha = \frac{1}{2}, \quad c \neq 0$$

$$P_n(an - cn^\alpha, (1 - a)n - cn^\alpha) \sim \eta(a, c) \frac{1}{n^{3/2 - 2\alpha}}, \quad 0 < \alpha < \frac{1}{2}, \quad c \neq 0$$

where

$$\eta(a, c) = \frac{c^2}{\sqrt{\pi(a(1 - a))^3}} \quad \text{and} \quad \kappa(a, c) = \exp \left[ \frac{-c^2}{a(1 - a)} \right]$$
Diagonal of $Q_n(\ast, \ast)$ vs. $P_n(\ast, \ast)$
Main Theorem for $Q_n(*,*)$, macro picture:

$$Q_n(an, bn) < \varepsilon^n, \quad 0 \leq a + b < 1, \quad \varepsilon = \varepsilon(a, b), \quad 0 < \varepsilon < 1$$

$$Q_n(an, bn) \sim v(a, b) \frac{1}{n^{3/2}}, \quad 1 < a + b < 2$$

$$Q_n(n, n) \sim \frac{1}{4}$$

where

$$v(a, b) = \frac{1}{\sqrt{32\pi} \ (2 - a - b)^{3/2} \ (1 - a - b)^{3/2}}$$
Main Theorem for $Q_n(*,*)$, micro picture:

$$Q_n(an-cn^\alpha,(1-a)n-cn^\alpha) < \varepsilon^{n^{2\alpha-1}}, \quad \frac{1}{2} < \alpha < 1, \quad \varepsilon = \varepsilon(a,b,\alpha), \quad 0 < \varepsilon < 1, \quad c > 0$$

$$Q_n(an-cn^\alpha,(1-a)n-cn^\alpha) \sim z(a) \frac{1}{n^{3/2-2\alpha}}, \quad \frac{3}{8} < \alpha < \frac{1}{2}, \quad c > 0$$

$$Q_n(an-cn^\alpha,(1-a)n-cn^\alpha) \sim z(a) \frac{1}{n^{3/4}}, \quad 0 < \alpha < \frac{3}{8}$$

$$Q_n(an+cn^\alpha,(1-a)n+cn^\alpha) \sim y(a,c) \frac{1}{n^{3/4}}, \quad \frac{3}{8} < \alpha < \frac{1}{2}, \quad c > 0$$

$$Q_n(an+cn^\alpha,(1-a)n+cn^\alpha) \sim w(c) \frac{1}{n^{3\alpha/2}}, \quad \frac{1}{2} < \alpha < 1, \quad c > 0$$

$$Q_n(n-cn^\alpha,n-cn^\alpha) \sim w(c) \frac{1}{n^{3\alpha/2}}, \quad 0 < \alpha < 1, \quad c > 0$$

where

$$w(c) = \frac{1}{16c^{3/2}\sqrt{\pi}}, \quad y(a,c) = \left(1 + \frac{\zeta(3/2)}{\sqrt{\pi}}\right) \frac{c^2}{\sqrt{\pi} a^{3/4}(1-a)^{3/2}},$$

$$z(a) = \frac{\Gamma(3/4)}{2^{3/4} \pi a^{3/4}(1-a)^{3/4}}$$
Proof idea:

**Lemma 1.** For \( j + k \leq n + 1, \)
\[
P_n(j, k) = B(n - k + 1, j) B(n - j + 1, k),
\]
where
\[
B(n, k) = \frac{n - k + 1}{n + k - 1} \binom{n + k - 1}{n}
\]
are the ballot numbers.

**Lemma 2.**
\[
Q_n(j, k) = \sum_{r=\max\{0, j+k-n-1\}}^{\min\{j,k\}-1} B(n - j + 1, k - r) B(n - k + 1, j - r) C_r
\]

Proof of the Main Theorem = Lemmas + Stirling’s formula + [details]
**Bijective combinatorics:**

123-avoiding permutations $\mapsto_{\text{RSK}}$ Pairs of SYT $\mapsto$ Dyck paths

**Corollary:** $P_n(i, j) = \text{Probability that random Dyck path is at height } j$

after $(i + j)$ steps

132-avoiding permutations $\mapsto$ Binary trees
5. Connections to Probability:

Random Dyck paths $\rightarrow$ Brownian excursion

*This explains everything!*

**Hint:**

1. heights in Dyck paths $\leftrightarrow$ distances to anti-diagonal in 123-av
2. tunnels in Dyck paths $\leftrightarrow$ distances to anti-diagonal in 132-av
6. Applications

**Corollary** [Miner-P.]

Let \( fp(\sigma) \) denote the number of fixed points in \( \sigma \in S_n \).

\[
\mathbb{E}[fp(\sigma)] \sim \frac{2\Gamma\left(\frac{1}{4}\right)}{\sqrt{\pi}} \cdot n^{\frac{3}{4}}, \quad \text{as } n \to \infty.
\]

where \( \sigma \in S_n \) is a uniform random 231-avoiding.

**Note:** For other patterns the expectations for the number of fixed points were computed by Elizalde (MIT thesis, 2004). Curiously, they are all \( O(1) \).

Main theorem also gives asymptotics for a large number of other statistics, such as rank, \( \lambda \)-rank, lis, last, etc.
2) Random permutation process:

\[ C_n = C_0 C_{n-1} + C_1 C_{n-2} + \ldots + C_{n-1} C_0 \]

Here \( k \) is chosen with probability 
\[ \rho_k = \frac{C_k C_{n-k-1}}{C_n} \propto k^{-3/2} (n - k - 1)^{-3/2} \]

**Question:** Can one define and compute the limit of this r.p.p.?
Bonus: final miracle

**Theorem** (Robertson, Saracino and Zeilberger, 2003; Elizalde, 2004, Elizalde and P., 2004)

The number of $132$-avoiding permutations with $k$ fixed points and $m$ excedances is equal to the number of $321$-avoiding permutations with $k$ fixed points and $m$ excedances.
7. The mysterious Baxter surface

*Baxter permutations:* Permutations $\sigma \in S_n$ such that there are no indices $i < j < k$ with $\sigma(j + 1) < \sigma(i) < \sigma(k) < \sigma(j)$ or $\sigma(j) < \sigma(k) < \sigma(i) < \sigma(j + 1)$.

$$B_n = \sum_{k=1}^{n} \frac{(n+1)\binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{1} \binom{n+1}{2}}{(n+1) \binom{n+1}{k+1} \binom{n+1}{2}}$$

*Note:* They are connected to tilings (Korn), to plane bipolar orientations (Bonichon – Bousquet-Mélou – Fusy), and 3-tuples of non-intersecting paths (Dulucq – Guibert, Fusy – Poulalhon – Schaefer). They were introduced in analytic context by Glen Baxter (1964).
Open Problem: What is the the limit shape of Baxter permutations?

Note: The bijections allow uniform generation, but don’t seem to be very helpful.

Note: Computation by Ted Dokos, UCLA.
Doubly alternating Baxter permutations

**Theorem** [Guibert–Linusson, 2000]
The number of Baxter permutations \( \sigma \in S_{2n} \) (or \( S_{2n+1} \)), such that both \( \sigma \) and \( \sigma^{-1} \) are alternating, is the Catalan number \( C_n \).

Denote by \( B_n \) the set of such permutations.

**Question:** What is the limit shape of permutations \( B_m \)?

Let \( P(m, i, j) \) denote the probability that a random \( \sigma \in B_{2m} \) has \( \sigma(i) = j \).
Theorem [Dokos–P., 2014]
Let \( 0 < \alpha < \beta < 1 - \alpha \). We have:
\[
P(m, \lfloor 2\alpha m \rfloor, \lfloor 2\beta m \rfloor) \sim \frac{\varphi(\alpha, \beta)}{m} \quad \text{as} \quad m \to \infty,
\]
where
\[
\varphi(\alpha, \beta) = \frac{1}{8\pi} \int_0^\alpha \int_0^{\alpha-y} \frac{dx\,dy}{(x+y)(\beta-x)(1-\beta-y)^{3/2}}.
\]
Final note: alternating permutations

Below is a plot of random $\sigma \in \text{Alt}_{500}$, i.e. $\sigma(1) > \sigma(2) < \sigma(3) > \sigma(4) < \ldots > \sigma(500)$. (only odd values are shown, boundary smoothened).

Right boundary is an inverted $\sin(x)$ curve, $0 < x < \pi/2$ [Diaconis–Matchett, 2012]

**Conjecture:** Limit shape of $\text{Alt}_n$ is horizontally flat.
Thank you!