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5/4/2020

RIPS

Asymptotics for Young tableaux

0) What this talk is about?

Objects: $f^\lambda = \#SYT(\lambda)$, $f^{\lambda/\mu} = \#SYT(\lambda/\mu)$

$K_{\lambda\mu} = \#SSYT(\lambda,\mu)$, $c^\lambda_{\mu\nu} = \#LR(\lambda/\mu,\nu)$

/Kostka and Littlewood-Richardson numbers/coeff/

$g(\lambda\mu\nu)$, $\bar{g}(\alpha\beta\gamma)$ $\leftarrow$ Kronecker coeff and reduced Kronecker

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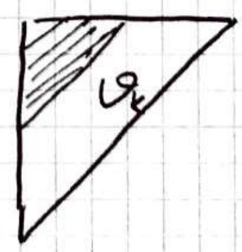
Questions:

- exact formulas
- asymptotics
- upper bounds

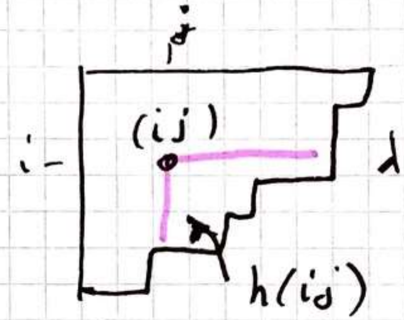
Ex $\vartheta_k = \delta_{2k}/\delta_k$



$f^{\vartheta_k} = ??$

1) Dimension

$\lambda = (\lambda_1, \lambda_2 \ldots) \vdash n$



(HLF) $\quad f^\lambda = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}$

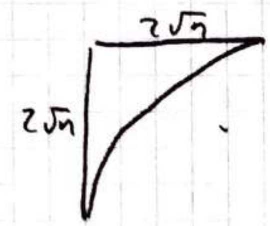
Th [Vershik-Kerov '85] $\quad D(n) := \max_{\lambda \vdash n} f^\lambda$

Then $\exists\, c_1, c_2 > 0$

$$\sqrt{n!}\, e^{-c_1\sqrt{n}} < D(n) < \sqrt{n!}\, e^{-c_2\sqrt{n}}$$

Conj [Vershik-$\underbrace{\text{Kerov-Pass}}_{1992}$] $\quad \exists \lim_{n\to\infty} \frac{1}{n} \log \frac{D(n)}{\sqrt{n!}} = c$

Th [Vershik-Kerov '81]



$f^\lambda > \sqrt{n!}\, e^{-o(n)} \implies \lambda$ has VKLS shape

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Notes: 1) next term in VKP-asy unlikely.

2) experiments show $c > h \leftarrow$ "typical" dim

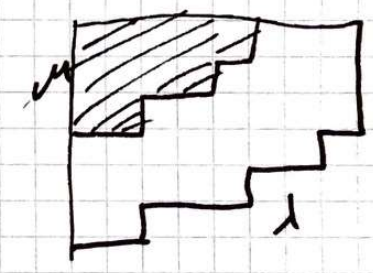
Conj. $\quad M(n) := \#\{\lambda \vdash n \; : \; f^{\lambda} = D(n)\}$

$M(n) = O(1)$ $\quad$ / even $e^{o(\sqrt{n})}$ is open /

2) Skew dim

$\lambda/\mu \vdash n \iff |\lambda| - |\mu| = n$

(JT) $\quad f^{\lambda/\mu} = n! \cdot \det\left(\dfrac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{i,j=1}^{\ell(\lambda)}$



Th [Morales-P-Panova] $\quad \lambda \sim n \cdot \alpha, \quad \alpha = (\alpha_1 \ldots \alpha_\ell), \; \alpha_i > \beta_i \geq 0$

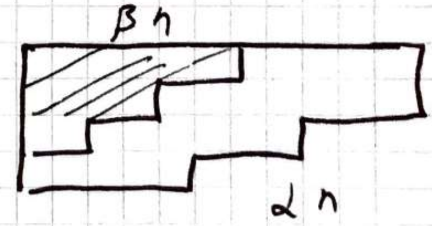
$\mu \sim n \cdot \beta, \quad \beta = (\beta_1 \cdots \beta_\ell)$

$\sum \alpha_i - \sum \beta_i = 1$

Then

$$\log f^{\lambda/\mu} = c(\alpha, \beta)\, n + o(n), \quad \text{where}$$

$$c(\alpha, \beta) = \sum_{i=1}^{\ell} (\alpha_i - \beta_i) \log(\alpha_i - \beta_i)$$



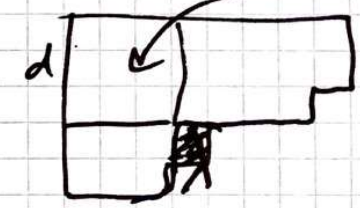
Note: same for all TVK $\longrightarrow$

Explatation

$$F(\lambda/\mu) := n! \prod_{(i,j)\in\lambda/\mu} \frac{1}{h_\lambda(i,j)}$$

Th [MPP] $\quad F(\lambda/\mu) \le f^{\lambda/\mu} \le \xi(\lambda/\mu)\cdot F(\lambda/\mu)$

Durfee sq



and

1) $\xi(\lambda/\mu) \le 2^n$

2) $\xi(\lambda/\mu) \le n^{2d^2}$

Th [Chan-P-Panova '20+]

$\forall \varepsilon > 0 \quad \forall \ell \ge 2 \quad \exists\, C_{\ell\varepsilon} > 0 \quad$ s.t. $\forall\, \lambda/\mu \vdash n$

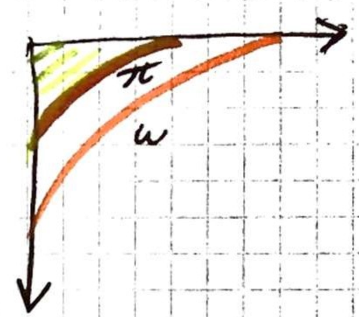
$\lambda_i - \lambda_{i+1} \ge \varepsilon n, \quad \lambda_\ell \ge \varepsilon n, \quad \ell(\lambda), \ell(\mu) \le \ell$

We have:
$$\frac{\phi(\lambda/\mu)}{C_{\ell\varepsilon}} \le \frac{f^{\lambda/\mu}}{F(\lambda/\mu)} \le C_{\ell\varepsilon}\, \phi(\lambda/\mu)$$

where
$$\phi(\lambda/\mu) = \prod_{1\le i<j\le \ell} \min\left\{\mu_i - \mu_j + j - i,\ \frac{\lambda_i + \ell - i}{\lambda_i - \lambda_j + j - i}\right\}$$

Stable shapes

$$\frac{1}{\sqrt{n}}\,[\lambda/\mu] \to \omega/\pi$$



Th [Morales-P-Panova + Morales-P-Tassy]

$$\exists \lim_{n\to\infty} \frac{1}{n} \log \frac{f^{\lambda/\mu}}{\sqrt{n!}} = c(\omega/\pi)$$

Theorem 3.3 (Weighted variational principle). *Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be a sequence of ASC, and let $\{w_n\}_{n\in\mathbb{N}}$ be a sequence of weight functions converging to a function ρ. Suppose that $\frac{1}{n}\gamma_n$ converges to a closed curved γ in $\mathbb{R}^3$ in the ℓ_∞ norm as $n\to\infty$. Then we have:*

$$\lim_{n\to\infty} \frac{1}{n^2} \log Z(H_{\gamma_n}, w_n) \;=\; \Psi(f_{\max}).$$

Here $f_{\max}: U \to \mathbb{R}$ is the only extension of γ in $\mathrm{Lip}_{[0,1]}$ which maximizes the following integral:

$$\Psi(f) \;:=\; \iint_U \Big(\sigma(\nabla f) + L(x_1,x_2,\nabla f)\Big)\,dx_1\,dx_2, \tag{3.4}$$

where U is the region enclosed by the projection of γ, and

$$L(x_1,x_2,\nabla f) \;:=\; \rho(x_1,x_2)\cdot(\partial_{x_1} f, \partial_{x_2} f, 1-\partial_{x_1} f - \partial_{x_2} f). \tag{3.5}$$

Moreover, for all $\epsilon > 0$, the height function of a random tiling chosen from the weighted measure associated to w_n on height functions with boundary γ_n, stays within ϵ of $f_{\max}$ with probability tending to 1 as $n\to\infty$.

Note $\lambda/\mu \in$ stable d-dim

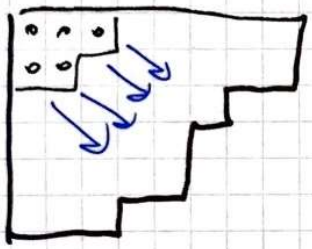
$$\Rightarrow \#SYT(\lambda/\mu) = (n!)^{\frac{d-1}{d}}\, e^{O(n)}$$

/ d-fixed, $n\to\infty$ /

Th [Naruse HLF, see MPP '15]

$$f^{\lambda/\mu} = n! \prod_{(ij)\in\lambda} \frac{1}{h_\lambda(ij)} \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(ij)\in D} h_\lambda(ij)$$

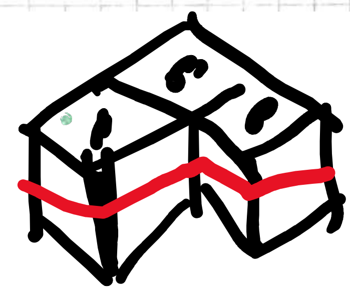
$\mathcal{E}(\lambda/\mu)$ = set of excited diagrams



Ex $\lambda=(332)$ $\mu=(21)$

$$\zeta(\lambda/\mu) = |\mathcal{E}(\lambda/\mu)|$$

Bounds on $\zeta(\lambda/\mu)$ $\Rightarrow$ MPP th for TVK



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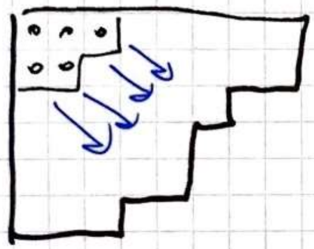
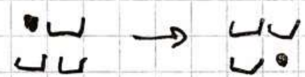
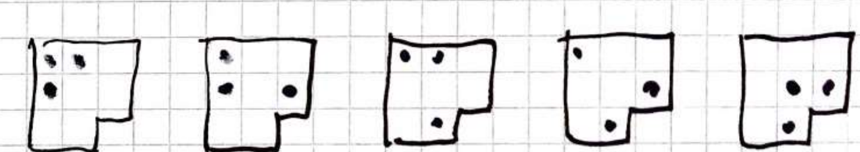




<u>Ex</u> $\lambda=(332)$ $\mu=(21)$



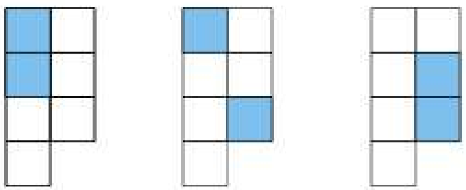
$\zeta(\lambda/\mu) = |\mathcal{E}(\lambda/\mu)|$

Bounds on $\zeta(\lambda/\mu) \Rightarrow$ MPP th for TVK

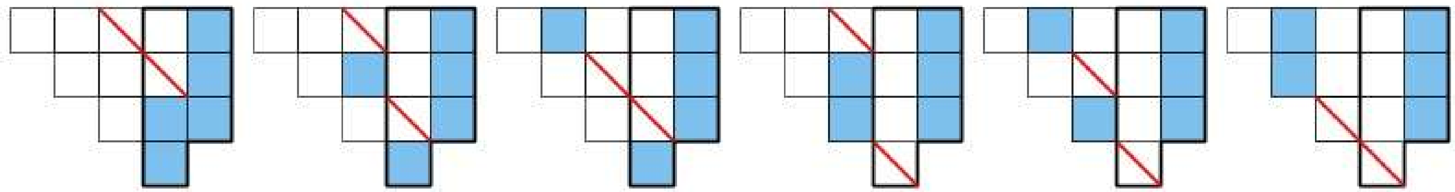
<u>Th</u> [Okounkov-Olshanski'98]

$$f^{\lambda/\mu} = n! \prod_{(ij)\in\lambda} \frac{1}{h(ij)} \sum_{D\in\mathcal{E}^{\wedge}(\lambda/\mu)} \prod_{(ij)\in\mathcal{E}^{\wedge}} (\lambda_i - i + j)$$

[Morales-Zhu]



(A)



(B)

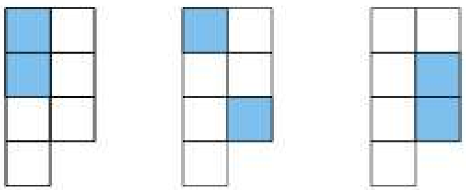
<u>Th</u> [Knutson-Tao'03]

puzzle version of [OB]

<u>Th</u> [OO, Stanley'03]

$\mu \leftarrow$ fixed $\qquad f^{\lambda/\mu} \sim C_{\alpha\beta}\, f^{\lambda}$, exact formula

$\lambda \leftarrow \alpha n$ $\qquad$ for $C_{\alpha\beta}$



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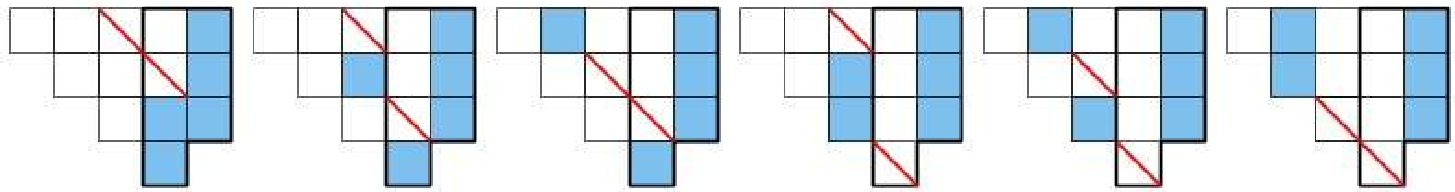


(B)

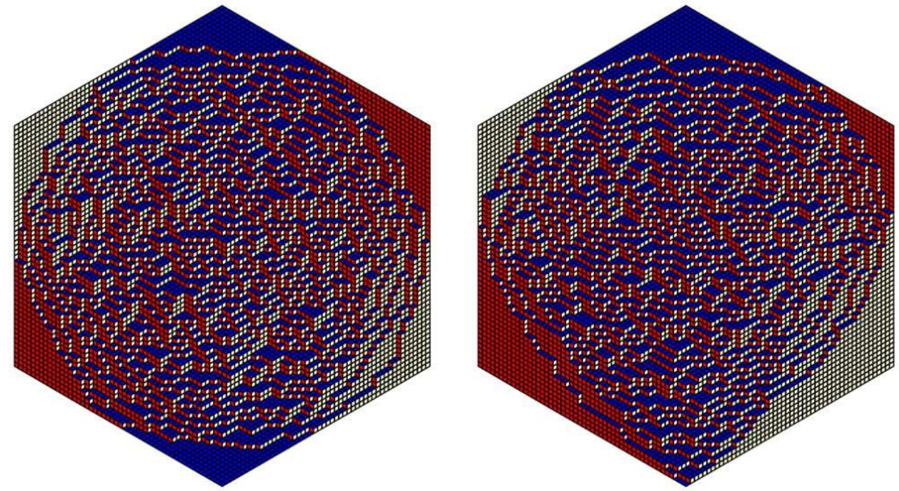
<u>Th</u> [Knutson-Tao'03]
puzzle version of [OO]

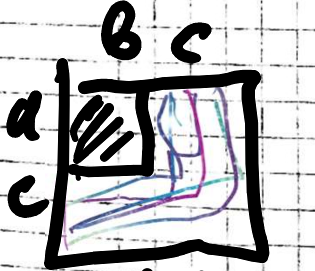
<u>Th</u> [OO, Stanley'03]
$\mu \leftarrow$ fixed $f^{\lambda/\mu} \approx C_{\alpha\beta}\, f^{\lambda}$, exact formula for $C_{\alpha\beta}$
$\lambda \leftarrow \alpha n$

<u>Bijection</u> [MPP] $\mathcal{E}(\lambda/\mu) \leftrightarrow$ Lozenge tilings

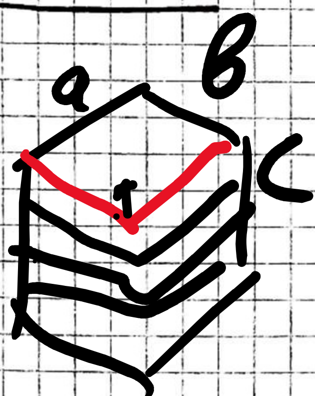
Example

$a = b = c = 50$

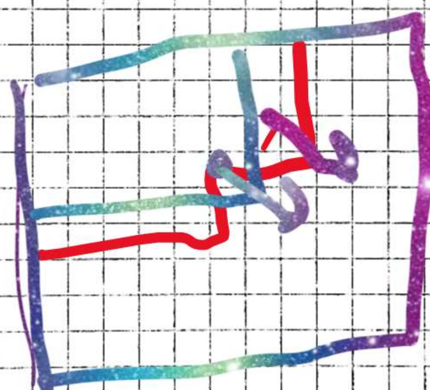




λ / μ



[BGR]



$\Omega(\lambda/\mu)$

Exact formulas

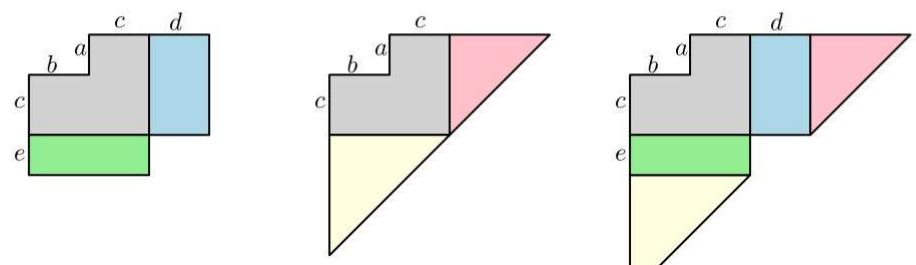
$f^{\lambda/\mu} = n! \cdot$ product of $\Phi(*)$ $\qquad \Phi(n) = 1!\,2!\cdots n!$

Corollary 1.1. *For all $a, b, c, d, e \in \mathbb{N}$, let λ/μ be the skew shape in* Figure 1 (i). *Then the number $f^{\lambda/\mu} = \left|\mathrm{SYT}(\lambda/\mu)\right|$ is equal to*

$$n!\,\frac{\Phi(a)\,\Phi(b)\,\Phi(c)\,\Phi(d)\,\Phi(e)\,\Phi(a+b+c)\,\Phi(c+d+e)\,\Phi(a+b+c+d+e)}{\Phi(a+b)\,\Phi(d+e)\,\Phi(a+c+d)\,\Phi(b+c+e)\,\Phi(a+b+2c+d+e)},$$

where $n = |\lambda/\mu| = (a+c+e)(b+c+d) - ab - ed$.

$$pp(a,b,c) = \frac{\Phi(a+b+c)\,\Phi(a)\,\Phi(b)\,\Phi(c)}{\Phi(a+b)\,\Phi(a+c)\,\Phi(b+c)}$$



(i) (ii) (iii)

Proof based on symmetric functions
+ identity