

L13
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Card Shuffling

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(1) Top to random

$$G = S_n, \quad R = \{1, (21), (231), \dots, (234 \dots n1)\}$$



Alg (for τ)

- * Wait until (n) goes to the top
- * Do 1 step. STOP

Th [Aldous-Diaconis]

τ -strong uniform

$$\text{mix} = D(E[\tau]) = O(n \log n)$$

L $\forall t$, the relative order of cards below (n) is uniform.

$\triangleright$ induction $\square$

$$L \Rightarrow Th \Rightarrow \text{mix} = n \log n + O(n)$$

$\Rightarrow$ cutoff

Def $\{X_t\} \leftarrow \text{r.w. on } \Gamma_n = \Gamma(G_n, S_n)$

has cutoff if $\forall \varepsilon > 0, \varepsilon < \frac{1}{2}$

$$\frac{t_{\varepsilon}^{(n)} - t_{1-\varepsilon}^{(n)}}{t_{\varepsilon}^{(n)}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

where

$$t_{\varepsilon}^{(n)} := \min \{ t : \text{sep}(t) < \varepsilon \}$$

↑
for r.w. on Γ_n

$$\text{sep}(t) = |G| \max_{y \in G} \left(\frac{1}{|G|} - Q^t(y) \right)$$

Ex 1) r.w. on $\mathbb{Z}_2^n$

2) r.w. on S_n (top to random)

Non-Ex $\mathbb{Z}_n$

(onj) Ex $\Gamma(S_n, \{(i, i') : 1 \leq i < i' \leq n\}) \leftarrow$ But has one in TV

(2) Star transpositions

$$G = S_n, \quad R_n = \{(1), (12), (13), \dots, (1n)\}$$

PROP $\text{diam } \Gamma(S_n, R_n) = \frac{3}{2}n + O(1)$

D $\sigma = (1, 3, 2, 5, 4, 7, 6, \dots, 2m+1, 2m)$

$$\text{dist}_\Gamma(1, \sigma) = 3m \quad \square$$

Th $\text{mix } \Gamma(S_n, R_n) = O(n \log n)$

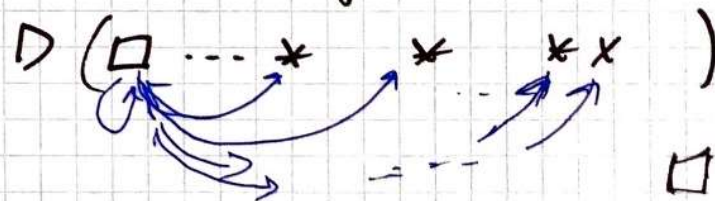
Alg (for π)

* Mark (n)

* If X_t at $\perp$ unmarked
and $(1i) \in R$, $i \in \{1\} \cup \text{marked}$,
mark it.

* STOP when all marked

$\triangle$ $\forall t$, relative order
of marked cards is
uniform.



$\Rightarrow \pi$ is strong
uniform.

(3)

$$\leq \mathbb{E}[\tau] = O(n \log n)$$

D 1) k marked cards, 1-st unmarked.

$$\mathbb{E}[\text{time until 1-st is marked}] = \frac{1}{(k+1)/n} = \frac{n}{k+1}$$

2) k marked cards, 1-st marked.

$$\mathbb{E}[\text{time until 1-st is unmarked}] = \frac{1}{(n-k)/n} = \frac{n}{n-k}$$

$$\begin{aligned} \mathbb{E}[\tau] &\leq \sum_k \left(\frac{n}{k+1} \right) + \left(\frac{n}{n-k} \right) \\ &\leq 2n \left(\sum_{k=1}^n \frac{1}{k} \right) = O(n \log n) \end{aligned}$$

□

$$D \text{ mix } \{X_t\} = O(\mathbb{E}[\tau]) = O(n \log n)$$

□

Note is No? cutoff (from this proof)

Lower Bound

$$\text{mix} = \Omega(n \log n)$$

D $\sigma = (23 \dots n1)$ has no fpts

By coupon collector's problem

not all i 's are used after $< n \log n - O(n)$ steps
w.h.p. $\square$

Markov scan

$$X_0 = 1, \quad X_{t+1} = X_t \cdot (t \bmod n, i), \quad i \in \{1 \dots n\} \text{ unif.}$$

Th [Mossel - Peres - Sinclair, 2004]

Markov scan has $\text{mix} = \Theta(n \log n)$

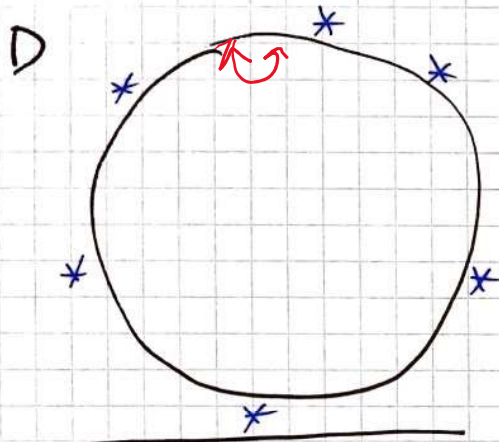
D Upper Bound $\leftarrow$ same strong unif. time argument
Lower Bound $\leftarrow$ hard. $\square$

$$(3) \quad G = S_n$$

$$R = \{ (12), (123, \dots, n)^{\pm 1} \}$$

$$\text{Prop/th mix} = O(n^3 \log n)$$

idea



* Mark cards as before
for top to random

* slow marking to $(\frac{1}{n!})$ pace

Th [Wilson]

$$\text{mix} = \Omega(n^3 \log n) \leftarrow \text{hard.}$$

Exc

$$\text{mix} = \Omega(n^3)$$

Cover Time

Def $\{X_t\} \leftarrow$ r.w. on $\Gamma = (V, E)$

$CT = \max(\text{over all starting } v \in V)$
time to visit all vertices.

Th [Matthews]

$$E[CT] = O(n \log n \text{ mix}) , n = |V|$$

D coupon collector's problem again! $\square$

Ex $\Gamma = \Gamma(\mathbb{Z}_2^m, R)$, $n = |\Gamma| = 2^m$, $\text{mix} = m \log m$

$$\begin{aligned} E[CT] &= O(n \log n \text{ mix}) = O(2^m \log 2^m m \log m) \\ &= O(2^m m^2 \log m) \\ &= \Omega(2^m) \end{aligned}$$

Two Bonus Parts

$$(4) \quad G = S_n, \quad R_n = \{1, (21), (321), (4321), \dots, (nn-1 \dots 1)\}$$

Th [Gates - Papadimitriou, 1979]

$$\text{diam } \Gamma(S_n, R_n) \leq \frac{5}{3}n + O(1)$$

/recently improved/

Open $\text{diam } \Gamma > (1+\varepsilon)n$??

$$(5) \quad x_{n+1} = ax_n + b \pmod{p}$$

prime number generator.

min time?

A $O(p \log p)$? no

$$\boxed{O(\log p \cdot \log \log p)}$$