

L12 (4/24/2020)

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Strong Uniform Times

Last time: $\{X_t\} \leftarrow \text{r.w. on } \Gamma(G, S)$

$\tau \leftarrow \text{stopping time}$ $\boxed{\tau}$

τ is strong uniform if

$$P(X_t = g \mid \tau = t) = \frac{1}{|G|} \quad \forall t > 0, \forall g \in G$$

Th [A-D]

$\tau \leftarrow \text{strong uniform}$

$$\Rightarrow \text{sep}(t) \leq P(\tau > t) \quad \forall t$$

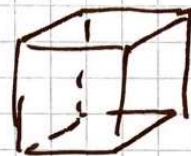
$$\text{where } \text{sep}(t) = |G| \max_{g \in G} \left(\frac{1}{|G|} - Q^t(g) \right)$$

Prop $\text{mix} \leq C \cdot E[\tau]$, where $\text{mix} := \min \{t : \text{sep}(t) \leq \frac{1}{2}\}$

Random walk on $\mathbb{Z}_2^m$ $\leftarrow$ hypercube

$$G = \mathbb{Z}_2^m, |G| = 2^m$$

$$S = \{(0, \dots, 1, \dots, 0), 1 \leq i \leq m\}$$



lazy random walk $X_0 = 1, X_{t+1} = X_t \cdot s_i^\epsilon, \epsilon \in \{0, 1\}$

$\{X_t\} \leftarrow$ choose a coord. i , flip a coin
if H, change coord. i

Th $\text{mix} = m \log m + O(m)$

$\triangleright$ Stopping time: mark coord. i when chosen
stop when all coord's are marked.

(0 ... * 0)

(0 ... * ... * 0)

(0 * * * 0)

$\vdots$

(* * * ... * *)

$\leq \tau$ is strong uniform $\square$

$$\Rightarrow \text{mix} \leq 2 \mathbb{E}[\tau] = 2m \log m + O(m)$$

coupon collector's problem



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Coupon collector's Problem

Setup: n coupons, collected one-by-one
w/ replacement (unif at random)

$X \leftarrow$ r.v. time to collect all n coupons

Th [Erdős-Rényi, 1961]

$$P(X < n \log n + cn) \rightarrow e^{-e^{-c}}$$

Prop

1) $IE[X] = n \log n + \gamma n + O(1)$

where $\gamma = 0.5772 \dots$ Euler-Mascheroni const

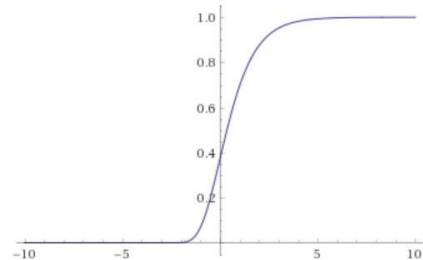
2) $Var(X) = \Theta(n^2)$

Cor $P(X > n \log n + \frac{\lambda}{n}) = O(\frac{1}{\lambda^2})$

Input interpretation:

plot $\exp(-\exp(-c))$ $c = -10$ to $+10$

Plot:



Proof of 1)

$X_k \leftarrow$ time to get k -th coupon

$$X = X_1 + \dots + X_n$$

$$E[X] = E[X_1] + \dots + E[X_n]$$

$$= \sum_{k=1}^n \frac{n}{n-k+1}$$

$$= n \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right)$$

$$= n \log n + \gamma n + O(1) \quad \square$$

$$\begin{aligned} X_k &= \text{Geo}(p_k), \quad p_k = \frac{n-k+1}{n} \\ E[X_k] &= \frac{n}{n-k+1}, \quad \text{Var}(X_k) = \frac{1-p_k}{p_k^2} \\ &\quad \uparrow 1/p_k \end{aligned}$$

$$\leq \frac{1}{(p_k)^2}$$

Proof of 2)

$$\text{Var}(X) = \text{Var}(X_1) + \dots + \text{Var}(X_n)$$

$$\leq \sum_{k=1}^n \left(\frac{n}{n-k+1} \right)^2$$

$$\leq n^2 \sum_{k=1}^n \frac{1}{k^2} = O(n^2) \quad \square$$

Trading coupons (Newman-Shepp, Erdős-Rényi) [1960] [1961]

$X^{(k)} \leftarrow$ time to collect k copies of each coupons

Th $P(X^{(k)} < n \log n + (k-1)n \log \log n + Cn) \rightarrow e^{-e^{-c}} \text{ as } n \rightarrow \infty \quad \forall k \leftarrow \text{fixed}$

Cor $IE[X^{(k)}] = n \log n + n \log \log n \times (k-1) + O(n)$

Random graphs applications $G(n, p)$ [1959]

Cor $P(G(n, p) \text{ - conn}) \rightarrow 0$ for $p < \frac{\log n}{n} = \frac{w(1)}{n}$ | $w(1) \rightarrow \infty$ as $n \rightarrow \infty$
 $P(G(n, p) \text{ has PM}) = 1$ —

Cor $P(G(n, p) \text{ is Hamiltonian}) \rightarrow 0$ for $p < \frac{\log n + \log \log n}{n}$

• • • $= \frac{w(1)}{n}$ • (5)

Back to r.w. on $\mathbb{Z}_2^n$

$T_n \leftarrow$ time to collect n coupons

$\{X_t\} \leftarrow$ lazy r.w. on $\mathbb{Z}_2^n$

We have $\text{sep}(t) \leq P(T_n > t)$

Note $P(X_t = (1 \dots 1)) = \frac{1}{2^n}$ $P(T_n < t) = \frac{1}{2^n} - \frac{P(T_n > t)}{2^n}$

$$\text{sep}(t) \geq 2^n \left(\frac{1}{2^n} - P(X_t = (1 \dots 1)) \right)$$
$$\geq P(T_n > t)$$

$$\Rightarrow \boxed{\text{sep}(t) = P(T_n > t)}$$

$$\Rightarrow \begin{cases} \forall \varepsilon > 0 & \exists c(\varepsilon) \text{ s.t.} \end{cases}$$

$$\begin{cases} \text{sep}(t) > 1 - \varepsilon \\ \text{sep}(t) < \varepsilon \end{cases}$$

$$t < n \log n - c(\varepsilon)n$$

$$t > n \log n + c(\varepsilon)n$$

Cutoff phenomenon

/cf. phase transition

⑥

Random walk on $(\mathbb{Z}_p)^n$

$p = \text{const}$

Idea: indep run stopping time $\tau \mathbb{Z}_p$
stop when all of these stopped. $\} \tau$

Claim $\tau \leftarrow \text{strong uniform.}$ $I \leftarrow \underline{\text{Exc.}}$

Cor 1 $\text{mix} = n \log n + O(p^2 n \log \log n) \mid \leftarrow \underline{\text{Exc}}$

Cor 2 $p = O(1) \Rightarrow \underline{\text{cutoff}}$ (phenomenon)

Note $\underline{n=1}$, $p \rightarrow \infty$ no cutoff

Random walk on $U(n, \mathbb{F}_2)$

$$U(n, \mathbb{F}_2) = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} : * \in \mathbb{F}_2 \right\} \quad R_n = \left\{ E_{ij} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ 0 & & & \ddots & \\ & & & & 1 \end{pmatrix} : 1 \leq i < j \leq n \right\}$$

$\{X_t\} \leftarrow$ lazy rw on $U(n, \mathbb{F}_2)$

Th $\text{mix} \leq n^2 \log n + O(n^2)$

Note $\text{mix} \geq \text{diam } \Gamma(U(n), R_n) = \Omega(n^2 / \log n)$

Proof: [strong uniform time $\approx$ construction]

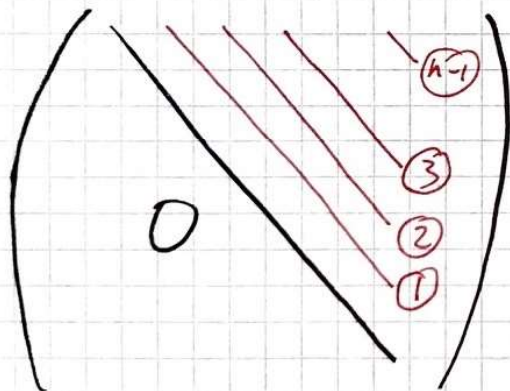
Def write $X_{t+1} = X_t \cdot (E_{ij})^\varepsilon$, $\varepsilon \in \{0, 1\}$ unif
stop when all (ij) are used.

Lemma τ defined above is strong uniform

$$D \quad X_t = (E_{i_1 j_1})^{\varepsilon_1} (E_{i_2 j_2})^{\varepsilon_2} \dots (E_{i_t j_t})^{\varepsilon_t}$$

$$/gh = hg \cdot [g, h] /$$

$$= \begin{pmatrix} E_{12}^{d_{12}} & E_{23}^{d_{23}} & \dots & E_{n-1, n}^{d_{n-1, n}} \end{pmatrix} \begin{pmatrix} E_{13}^{d_{13}} & E_{24}^{d_{24}} & \dots \end{pmatrix} \cdot (*)_{[12]} \\ \cdot \begin{pmatrix} E_{14}^{d_{14}} & E_{25}^{d_{25}} & \dots \end{pmatrix} (*)_{[123]} \dots (*)_{[1 \dots n]}$$



$\Rightarrow$ coupon collectors problem
on $\{(i, j), 1 \leq i < j \leq n\}$

$$D \quad \mathbb{E}[\tau] = \binom{n}{2} \log \binom{n}{2} + O(\binom{n}{2}) \\ = n^2 \log n + O(n^2)$$