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Random walks

$G = \langle S \rangle$ – fin. group, $S = \{s_1, \dots s_k\}$, $S = S^{-1}$

$$\begin{cases} X_0 \leftarrow 1 \\ X_{t+1} \leftarrow X_t \cdot s_i, \quad s_i \in S, \text{ unif at random} \end{cases}$$

$\{X_t\} \leftarrow$ random walk on $\Gamma(G, S)$

$Q^t(g) = P(X_t = g) \leftarrow$ distr. of $\{X_t\}$ after t steps

Prop $\forall\, S = S^{-1}$, $1 \in S$

$$Q^t(g) \to \frac{1}{|G|} \quad \forall g \in G$$

← Exc

Note: this means $\{X_t\}$ is ergodic

Mixing time

Def 1 $\quad mix_1 := \min\{t : |Q^t - U|_{TV} \le \frac{1}{4}\}$, $U(g) = \frac{1}{|G|}$

$$|P - Q|_{TV} := \frac{1}{2}|P-Q|_{\ell_1} := \frac{1}{2}\sum_{g \in G} |P(g) - Q(g)|$$

$$:= \max_{A \subseteq G} |P(A) - Q(A)| \qquad \| \leftarrow \text{Exc}$$

Note: if $\frac{1}{4} \leftrightarrow (\varepsilon < \frac{1}{4}) \Rightarrow mix_1' < C(\varepsilon)\, mix_1$ $\quad \|$ Exc

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Def 2 $\quad mix_2 := \min\{t : sep(Q^t, U) \le \frac{1}{2}\}$

$$sep(Q^t, U) := |G| \max_{g \in G} \left(\frac{1}{|G|} - Q^t(g)\right)$$

$$sep(Q^t, U) < \varepsilon \iff Q^t(g) > \frac{1-\varepsilon}{|G|} \quad \forall g \in G$$

Note: Our main def for today.

Prop

(1) $sep(t+1) \le sep(t)$

(2) $sep(t+s) \le sep(t)\, sep(s)$

(3) $sep(t) \sim c\, \varrho^t$ as $t \to \infty$, $0 \le \varrho \le 1$

D of (1): $sep(t) = \varepsilon \Rightarrow Q^t = (1-\varepsilon) U + \varepsilon N$ ← "noise"

$$Q^{t+1} = Q^t * \underbrace{Q^1}_{=P} \quad \text{where} \quad Q^1(g) \doteq \begin{cases} 1/|S|, & g \in S \\ 0, & \text{oth.} \end{cases}$$

$$\Rightarrow Q^{t+1} = \big((1-\varepsilon) U + \varepsilon N\big) * P$$

$$= (1-\varepsilon) U * P + \varepsilon N * P$$

$$= (1-\varepsilon) U + \varepsilon (N * P) \qquad \blacksquare$$

of (2) $Q^t = (1-\varepsilon) U + \varepsilon N_1$, $Q^s = (1-\delta) U + \delta N_2$

$$Q^{t+s} = Q^t * Q^s = \big((1-\varepsilon) U + \varepsilon N_1\big) * \big((1-\delta) U + \delta N_2\big)$$

$$= (1-\varepsilon\delta) U + \varepsilon\delta (N_1 * N_2)$$

$$\Rightarrow sep(t+s) \le \varepsilon\delta = sep(t)\, sep(s) \qquad \blacksquare$$

(3) $A = (a_{gh})_{|G|\times|G|}$, $a_{gh} = \begin{cases} 1/|S| , & hg^{-1}\in S \\ 0 , & \text{oth} \end{cases}$

Obs $Q^t(g) = A^t \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}_g = \frac{1}{|G|}\underbrace{\begin{pmatrix}1\\ \vdots\\1\end{pmatrix}}_{=v_0} + \lambda_1^t v_1 + \dots + \lambda_{|G|-1}^t v_n$

$\{(v_i,\lambda_i)\}$ ← eig. vectors, eig. values of A

as $t\to\infty$ $Q^t(g) = \frac{1}{|G|} + (\rho^t\cdot c) + o(\rho^t c)$

$\Rightarrow \text{sep}(t) \sim \rho^t c'$ ∎

Exc $\exists\, c_1, c_2 > 0$ s.t. $c_1(\text{mix}_1) < (\text{mix}_2) < c_2(\text{mix}_1)$

Q: how do you bound mix ← mix_2 ??

Prop $\text{mix} < C\left(\frac{1}{1-\lambda_1}\right)\log|G|$ ∎ | ← Exc

Ex $G = \mathbb{Z}_m$, $S = \{\pm 1\}$

$$A = \begin{pmatrix} 0 & 1/2 & & & 1/2 \\ 1/2 & 0 & 1/2 & & \\ & 1/2 & 0 & 1/2 & \\ & & 1/2 & 0 & \ddots \\ & & & \ddots & \ddots & 1/2 \\ 1/2 & & & & 1/2 & 0 \end{pmatrix}$$

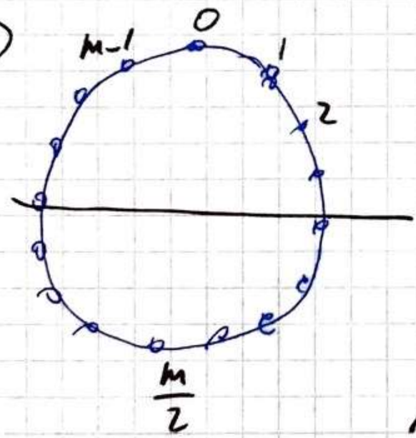
Exc $\lambda_j = \cos(2\pi j/m)$

$$\Rightarrow \lambda_0 = 1, \quad \lambda_1 = 1 - \frac{(2\pi)^2}{m^2} + \cdots$$
$$= 1 - \frac{c}{m^2} + O\left(\frac{1}{m^4}\right)$$

$\Rightarrow$ $\text{mix} = O(m^2 \log m)$ | **Th** $\text{mix} = \Theta(m^2)$

Prop $\text{mix} = \Omega(m^2)$

D



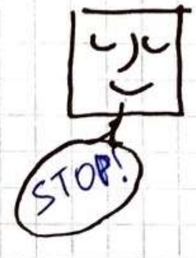
after t step $X_t \in [-c\sqrt{t}, +c\sqrt{t}]$ w.h.p.

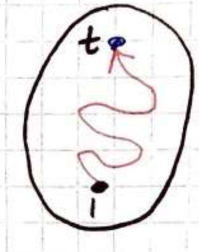
$\Rightarrow$ to reach $\frac{m}{2}$ w/ $Q^t(\frac{m}{2}) \geq \frac{1}{2m}$

need $\Omega(m^2)$ steps $\blacksquare$

Note for TV dist $Q^t(\text{bottom half}) \geq \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ $\blacksquare$

Strong uniform time





Def τ ← (prob.) stopping time

is called strong uniform if

$$P(X_t = g \mid \tau = t) = \frac{1}{|G|} \qquad \forall g \in G \quad \forall t \geq 0$$

Th [Aldous-Diaconis, 1987]

(1) $G = \langle S \rangle$, $S = S^{-1}$, $1 \in S$, τ ← strong unif.

Then $sep(t) \leq P(\tau > t) \quad \forall t$

(2) $\exists \tau$ s.t. $=$ $\forall t$.

Cor $mix < 2 \cdot \mathbb{E}[\tau]$

D By Markov ineq $P(\tau \geq 2\mathbb{E}[\tau]) \leq \frac{1}{2}$

$$sep(2\mathbb{E}[\tau]) \leq P(\tau > 2\mathbb{E}[\tau]) \leq \frac{1}{2} \qquad \blacksquare$$

Proof of (1)

$$Q^t = \sum_{k \le t} U \cdot P(\tau = k) + N\, P(\tau > t)$$

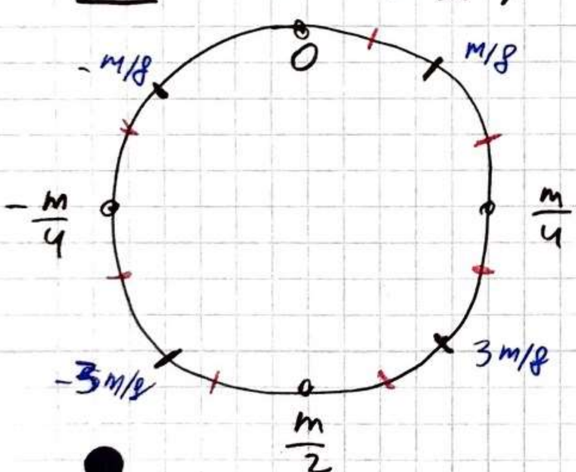
$$= U \cdot P(\tau \le t) + N\, P(\tau > t)$$

$$\Rightarrow \quad sep(Q^t, U) \le P(\tau > t) \qquad \blacksquare$$

Proof of (2) | ← **Exc**

clue: can flip coins (!)

Ex $\quad G = \mathbb{Z}_m, \quad S = \{0, 0, 1, -1\}, \quad m = 2^r$



Alg $\quad \boxed{\tau}$

$$\begin{cases} \text{walk until } \pm m/4 \\ \text{— // —} \quad \pm m/4 \pm m/8 \\ \text{— // —} \quad \pm m/4 \pm m/8 \pm m/16 \\ \dots \\ \text{wal until odd.} \\ \text{one more step} \end{cases}$$

L τ is strong uniform.

D (1) either of $\pm\frac{m}{4}$ is unif <u>cond.</u> t

(2) — // — $\pm\frac{m}{4}\pm\frac{m}{8}$ — // —

⋮

(r-1) — // — odd — // — // —

(r) <u>all</u> unif. <u>cond. t</u> $\blacksquare$

$$\Rightarrow \quad Mix \le 2\,\mathbb{E}[\tau] = c\left(\frac{m}{4}\right)^2 + c\left(\frac{m}{8}\right)^2 + c\left(\frac{m}{16}\right)^2 + \dots$$

$$\le c\left(\frac{m}{4}\right)^2\left(1 + \frac{1}{4} + \frac{1}{16} + \dots\right)$$

$$\le \underline{\underline{c' m^2}} \qquad \blacksquare$$

Random walk on $\mathbb{Z}_2^m$ ← hypercube

$G = \mathbb{Z}_2^m$, $|G| = 2^m$

$S = \{ (0 \ldots \underset{i}{1} \ldots 0),\ 1 \le i \le m \}$

lazy random walk $X_0 = 1$, $X_{t+1} = X_t \cdot s_i^{\varepsilon}$, $\varepsilon \in \{0,1\}$

$\{X_t\}$ ← choose a coord. i, flip a coin
if H, change coord. i

Th $\text{mix} = m \log m + O(m)$

▷ Stopping time: mark coord. i when chosen
stop when all coord's are marked.

$(0 \ldots * \quad 0)$

$(0 \ldots * \ldots * 0)$

$(0 * * \quad * 0)$

$\vdots$

$(* * * \cdots * *)$

L τ is strong uniform □

$\Rightarrow \text{mix} \le 2\, \mathbb{E}[\tau] = 2 m \log m + O(m)$

coupon collector's problem