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Cayley graphs

$G = \langle S \rangle$, $S = \{s_1, \ldots s_k\}$, $S = S^{-1}$ / $\Leftrightarrow$ $\forall s \in S$, $s^{-1} \in S$ /

$\Gamma = \Gamma(G, S) \leftarrow$ Cayley graph

vertices $g \in G$

edges (g, gs), $g \in G$, $s \in S$

Questions about Cayley graphs

(1) $\mathrm{diam}(\Gamma) = ??$

(2) $\mathrm{mix}(\Gamma) = ??$

(3) $\mathrm{eig}(\Gamma) = ??$

¿expander?

(4) Γ - planar?

(5) Γ - edge-transitive?

(6) Γ - Hamiltonian?

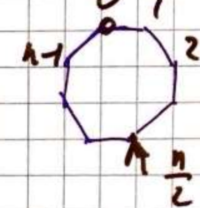
...

Today: some overview before (1) → (2) → (3) direction

Note $\mathrm{diam}(\Gamma) = \max_{x,y \in G} |x,y|_\Gamma$

Diameter of Cayley graphs

(0) $S = G \Rightarrow \Gamma(G, G) = K_{|G|}$, so $\operatorname{diam} \Gamma = 0$



(1) $G = \mathbb{Z}_n$, $S = \{\pm 1\}$, $\operatorname{diam} \Gamma(G, S) = \left[\frac{n}{2}\right]$

(2) general graphs $G = (V, E)$, $|V| = n$

$$\operatorname{diam} = O\left(\frac{\log n}{\log\log n} \cdot r(G)\right), \quad r(G) = \sum_{v \in V} \frac{1}{\deg(v)}$$

[Erdős Pach Spencer 1988]

for us $\deg(v) = |S|$, so $r(\,) = |G|/|S| \to$ awful.

(3) $G = S_n$, $G = \langle R \rangle$

Prop 1 $R_n = \{(12), (12 \cdots n)^{\pm 1}\} \Rightarrow \operatorname{diam} \Gamma(S_n, R_n) = \Theta(n^2)$

D we'll give one later using LA $\square$

Prop 2 $R_n = \{(12), (23), \ldots (n-1, n)\} \Rightarrow \operatorname{diam} \Gamma(S_n, R_n) = \binom{n}{2}$

D $\operatorname{dist}(1, \sigma) \leq \operatorname{dist}((123 \ldots n), (n\, n-1, \ldots 1)) = \binom{n}{2}$
$= \operatorname{inv}(\sigma)$

Conj $\forall R_n$, $\langle R_n \rangle = S_n$, $\operatorname{diam}(S_n, R_n) = O(n^2)$
/ say $\le 3n^2$ /

Th [Babai, Seress, 1988]
$$\operatorname{diam}(S_n, R_n) = e^{O(\sqrt{n \log n})}$$

Th [Helfgott-Seress, 2014] ← Annals Math
$$\operatorname{diam}(S_n, R_n) = e^{O((\log n)^4 \log\log n)}$$

(4) Lower Bounds

Prop $\operatorname{diam} \Gamma(G,S) \geq \log_{|S|} |G| - 1$

$\square$ # $g \in G$ at dist $0 \leftarrow 1$
-"- $1 \leftarrow |S|$
-"- $2 \leq |S|^2$
- - - -
$\operatorname{diam} \leq |S|^{\operatorname{diam}}$

$$1 + |S| + |S|^2 + \ldots + |S|^{\operatorname{diam}} \geq |G|$$
$\blacksquare$

Q: what about S_n?

Th $\exists R_n, |R_n| = O(1), \langle R_n \rangle = S_n$

s.t. $\operatorname{diam} \Gamma(S_n R_n) = O(\log n!) = O(n \log n)$

D **Let** $n = p+1$, p - prime

$S_n \leftrightarrow$ perm's of $\{0, 1, 2, \dots, p-1, \infty\} \in P\mathbb{F}_p$

$$\begin{array}{lll|l}
\alpha: & a \to a+1 & (p) & (1, 2, 3, \dots, 0, \infty) \\
\beta: & a \to 2a & (p) & (0, 2, 4, \dots, p-2, \infty) \\
\gamma: & a \to \frac{1}{a} & (p) & (\infty, 1, \dots, 0)
\end{array} \quad R_n = \{\alpha^{\pm 1}, \beta^{\pm 1}, \gamma\}$$

L1 $\operatorname{dist}_\Gamma(i\,j) = O(\log n)$, $\Gamma = \Gamma(S_n, R_n)$

D
$$\begin{cases} i \xrightarrow[\alpha^{-1}, \beta^{-1}]{} \dots \longrightarrow 1 \xrightarrow[\alpha^{-1}]{} 0 \xrightarrow[\gamma]{} \infty & O(\log n) \text{ steps} \\ j \to \dots \to 0 & \text{— // —} \end{cases}$$

$\underline{\gamma :} \ \langle i \leftrightarrow j \rangle$

Repeat in reverse ▨

L2 $R_n = \{(i\,j), 1 \le i < j \le n\}$

$\operatorname{diam}(S_n, R_n) = O(n)$ ▨ Exc

▨

(5) Linear groups

Conj [Babai] $G = PSL(n,q)$, n fixed, $q \to \infty$

$$\operatorname{diam} \Gamma(G, R) = O_c(\log|G|)^c, \quad c = c(n)$$

Note: mostly proved by now (Bourgain, Gamburd, Breuillard, Green, Tao, etc)

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(6) $U(n,q)$

$q \leftarrow$ fixed (say $q=2$)

$$G_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \; * \in \mathbb{F}_q \right\} \qquad R_n = \left\{ \begin{array}{c|c} & j \\ \hline i & \begin{pmatrix} 1 & & & \\ & \ddots & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \end{array} = E_{ij}, \; 1 \le i < j \le n \right\}$$

Prop $\operatorname{diam}\Gamma(G_n, R_n) = O(n^2)$

$\triangleright$ each row $= O(n)$ ▨

Lower Bound: $\operatorname{diam}\Gamma \geq \log_{\binom{n}{2}} 2^{\binom{n}{2}} = \dfrac{\log 2^{\binom{n}{2}}}{2\log n} = c\,\dfrac{n^2}{\log n}$

Q/Conj $\operatorname{diam} = \Theta(n^2)$

Th [Ellenberg, 1993]

$G_n = U(n,2)$, $R_n = \{E_{i\,i+1},\ 1 \le i \le n-1\}$

Then $\operatorname{diam} \Gamma(G_n, R_n) = \Theta(n^2)$

D **UB**

$$A = \begin{pmatrix} 1 & * & * & * & * & * & * \\ & 1 & & & & & \\ & & 1 & & & & \\ & 0 & & 1 & & & \\ & & & & 1 & & \\ & & & & & & 1 \end{pmatrix}$$

Case 1 $(1\,1\,*\,*\,\ldots\,*)$
$(1\,*\,*\,\ldots\,*) \uparrow +$

Case 2 $(1\,0\,*\,*\ \ *)$
$(1\,0\,0\ \ 0) \uparrow$] **Now**

$a(n) = \operatorname{dist}(1, A) \le (n-1) + a(n-1) + a(n-1)$ ← terrible(!)
$\le e^{O(n)}$

$a(n) \le a(n-1) + \underbrace{O(1)}_{2}$ V | cleanup idea

$\le 2n$

$\Rightarrow \operatorname{diam} \le a(1) + \ldots + a(n) = O(n^2)$ ◭

LB Need to improve trivial $\Omega(n^2/\log n)$ to $\Omega(n^2)$

Heaps of pieces

Alphabet: $A = \{a_1, \dots, a_k\}$, $k = n-1$

Words: $A^* = \{a_{i_1} a_{i_2} \cdots a_{i_\ell}\}$

Commutativity relations $a_i a_j = a_j a_i \quad \forall\ |i-j| \geq 2$

Main Lemma: $\underbrace{\left(\#\text{ words of length} \leq \ell \ /\text{modulo c.r.}/\right)}_{=\, w(\ell)} \leq c^\ell$ some $c>1$

Ex $k=3$

ℓ	1	2	3	4	...	ℓ
$w(\ell)$	3	8	21	55	...	?

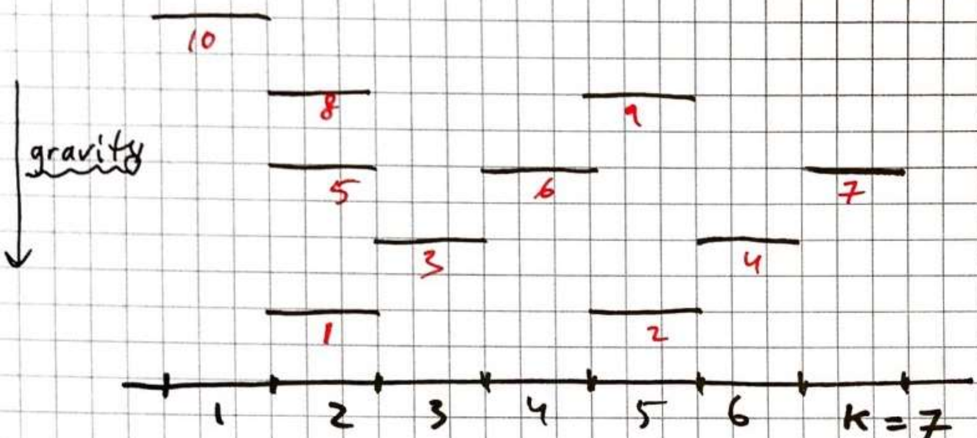
OEIS A001906

$a_1 a_3 = a_3 a_1$

$a_1 a_1 a_3 = a_1 a_3 a_1 = a_3 a_1 a_1$

$w(\ell) = 3\, w(\ell-1) - w(\ell-2)$

Heaps visualized



$\Longleftrightarrow$ $a_2 a_5 a_3 a_6 a_2 a_4 a_7 a_2 a_5 a_1$

word in $A = \{a_1 \ldots a_7\}$

Def word is *simple* if all letters commute.

Ex $a_2 a_5 a_7$

Th (Viennot, 1986)

$\Leftarrow$ (Cartier-Foata, 1969)

$$\sum_{w \in \text{words in } A/CR} w = \left(\sum_{\substack{w \in \text{simple words} \\ \text{in } A}} (-1)^{|w|} w \right)^{-1}$$

Ex $k = 3$

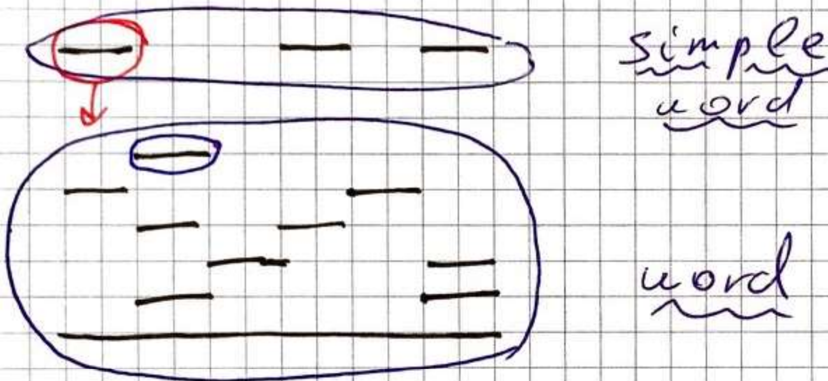
$$\sum \omega(\ell) t^\ell = \frac{1}{1 - 3t + t^2}$$

simple words:

ϕ, a_1, a_2, a_3, $a_1 a_3$

D (visual proof)

$$\left(\sum \text{words}\right)\left(\sum \pm \text{simple words}\right) = 1$$



Idea

1) find smallest movable piece

2) move ↓ if possible, ↑ otherwise.

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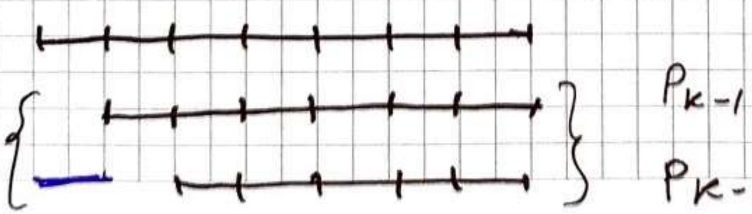
only $\emptyset$ word remains ∎

Def $$P_k(t) := \sum_{w \in \text{simple word}} (-t)^{|w|}$$

Prop (1) $P_k(t) = P_{k-1}(t) - t\,P_{k-2}(t)$

(2) $$P_k(t) = \sum_{m=0}^{\lfloor k/2 \rfloor} (-1)^m \binom{k-m+1}{m} t^m$$

D



∎

$$\begin{cases} P_0(t) = 1 , \quad P_1(t) = 1 - t \\ P_k = P_{k-1} - t\, P_{k-2} \end{cases}$$

$$\Rightarrow P_k(t) = \frac{1}{2}\left(1 + \frac{1-2t}{\sqrt{1-4t}}\right)\left(\frac{1+\sqrt{1-4t}}{2}\right)^k + \frac{1}{2}\left(1 - \frac{1-2t}{\sqrt{1-4t}}\right)\left(\frac{1-\sqrt{1-4t}}{2}\right)^k$$

$z \leftarrow \sqrt{1-4t}$

$$= \frac{1}{2^{k+2}\, z}\left[(1+z)^{k+2} - (1-z)^{k+2}\right]$$

roots, $1 \le r \le n+1$

$$i \tan\left(\frac{\pi r}{n+2}\right)$$

calculation

/Exc/

$\Rightarrow$ roots of $P_k(t)$ are $\geq \frac{1}{4}$

$$\Rightarrow \sum_{w \text{ words}} t^{|w|} = \sum w_k(\ell)\, t^\ell = \frac{1}{P_k(t)}$$ has poles at $\geq \frac{1}{4}$

$$\Rightarrow w_k(\ell) \leq 4^\ell (k+2)$$

$\blacksquare$

Back to $U(n,2)$

$U(n,2) = \langle E_i \,,\ 1 \le i \le n-1 \rangle$

$k = n-1$

Improved LB:

$\#$ words in E_i of length $\ell \le w_k(\ell) \le 4^{\ell(k+2)}$

$\le 4^{\ell(n+1)}$

$$\Rightarrow \operatorname{diam} \Gamma(U(n,2), \{E_i\}) \ge c \log \left[2^{\binom{n}{2}} / n+1 \right]$$

$$= \Omega(n^2)$$

$\blacksquare$

Ex $\quad G = S_n, \quad R = \{(12), \dots (n-1, n)\}$

LB $= \dfrac{\log n!}{\log n} = \Theta(n)$

Improved LB $\qquad \log n! = \Theta(n \log n)$

True $\qquad \Theta(n^2)$