HOME ASSIGNMENT 2 (MATH 218, FALL 2021)

1) Prove that for every $k \geq 1$ there exists a sequence $\{p_n\}$ such that

$$\mathbb{P}\big[\text{graph } G(n,p_n) \text{ has a connected component with } k \text{ vertices}\big] \to 1 \quad \text{as } n \to \infty.$$

2) Let $v_i = (x_i, y_i) \in \mathbb{Z}^2$, where $1 \leq i \leq n$, $n > 10^{10}$, and $|x_i|$, $|y_i| \leq 1.4^n$. Prove that there exists $A, B \subset [n]$, $|A| + |B| \geq 1$, such that

$$\sum_{i \in A} v_i = \sum_{j \in B} v_j$$

3) Prove that there is a constant $\varepsilon > 0$ such that there exist $S \subset \{0,1\}^n$ with $|S| > (1+\varepsilon)^n$ and every $a, b \in S$ differ in at least $n/4$ coordinates, for all n.

4) Let $G = (V, E)$ be a bipartite graph on $|V| = n$ vertices, and suppose there is a subset $C_v \subset [N]$ of colors assigned to each $v \in V$. Suppose $|C_v| \geq \log_2 n$. Prove that there is a proper coloring of graph G, where each vertex gets a color in C_v.

5) Let $G := G(n, 1/2)$ be a random graph on $n = 2k$ vertices. Prove that the probability that maximal degree in G is $\leq k-1$ is at least $1/2^n$.

6) Let Let $g_1, \ldots, g_n \in G$ be uniform random i.i.d. elements in a finite abelian group G. Prove that $\langle g_1, \ldots, g_n \rangle = G$ with probability $> 1 - \varepsilon$, for $n = \log_2 |G| + O_\varepsilon(1)$, where the constant can depend on $\varepsilon > 0$. Can you extend the result to *nonabelian* groups?

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This Homework is due Wednesday Nov. 3, at 2:59 pm (right before class). Please upload your solution to the Gradescope. Please read the collaboration policy on the course website. Make sure you write your name in the beginning and your collaborators' names at the end.

P.S. Each item above has the same weight.