HOME ASSIGNMENT 1   (MATH 218, FALL 2021)

1) In the coupon collector's problem with n coupons, show that time to collect the first $n/2$ coupons is $O(n)$. Let τ be the time to collect all n coupons. In class, we showed that $\mathbb{E}[\tau] = n\log + O(n)$. Give a probabilistic estimate of the number k of coupons that collected exactly once at time τ. Prove that the expected time to collect each of those k coupon at most $O(n\log k)$. Conclude that for the time τ' to collect each coupon twice, we have:

$$\mathbb{E}[\tau'] = n\log n + O(n\log\log n).$$

2) Place m black pawns and m white knights randomly on a $m \times m$ chessboard (without two pieces at the same square). Denote by X_m the number of ways one of the knights can attack one of the pawns.[1] Compute $\mathbb{E}[X_m]$ and the limit of $\mathbb{E}[X_m]$ as $m \to \infty$.

3) Let $A \subset \mathbb{R}^2$ be a set of 10 points in the plane. Prove that A can be covered by a union of 10 disjoint unit discs.

4) Let $S = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$ be a unit sphere. Two points $p, q \in S$ are called *orthogonal* if $Op \perp Oq$. Let $A \subset S$ be a measurable subset of points with no two orthogonal points. Prove that

$$\frac{\operatorname{area}(A)}{\operatorname{area}(S)} \leq \frac{1}{3}$$

Find a set A as above, with the area ratio $> 1/4$.

5) Prove that there exist an absolute constant $\alpha > 0$ such that the following property holds. Let M be an n by n matrix with pairwise distinct entries. Then there is a permutation of the rows of M so that no column in the permuted matrix contains an increasing subsequence of length $> \alpha\sqrt{n}$.

6) Suppose $k \geq 4$ and let H be an k-uniform hypergraph with at most $4^{k-1}/3^k$ edges. Prove that there exists a coloring of the vertices of the hypergraph H with colors $\{0, 1, 2, 3\}$, so that every edge has all four colors.

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This Homework is due Wednesday Oct 13, at 2:59 pm (right before class). Please upload your solution to the Gradescope. Please read the collaboration policy on the course website. Make sure you write your name in the beginning and your collaborators' names at the end.

P.S. Each item above has the same weight.

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[1] Read about moves of chess pieces on `https://en.wikipedia.org/wiki/Chess` if these are unfamiliar.