HOME ASSIGNMENT 2 (MATH 206A, WINTER 2024)

I. Let $I_i = [a_i, b_i] \subset \mathbb{R}$, $1 \le i \le n$ be a family of intervals. Denote by $P = ([n], \prec)$ a poset on these intervals with $I_i \prec I_j$ if $b_i < a_j$. These are called *I-posets* and intervals $\{I_i\}$ are their representations. Prove that $P = (X, \prec)$ is an I-poset if an only if for all $x \prec y$, $u \prec v$ we have $x \prec v$ or $u \prec y$.

II. Prove or disprove:
a) Every I-poset has a representation with intervals of lengths in $[1, 3]$.
b) Every I-poset has a representation with intervals of at most 3 distinct lengths.

III. Let n be even. A subset $A \subseteq [n]$ is called *nice* if the numbers of even and odd elements in A differs by at most 1. Suppose a collection of subsets $\mathcal{A} \subseteq 2^{[n]}$ satisfies the property

$$A \subset B,\ A, B \in \mathcal{A} \quad \Longrightarrow \quad B \smallsetminus A \text{ is nice.}$$

Prove that $|\mathcal{A}| \le \binom{n}{n/2}$.

IV. A chain in a finite poset $P = (X, \prec)$ is called *maximal* if it is contained in no other chain. Prove that the maximum number of disjoint maximal chains is equal to the minimum size of a set intersecting all maximal chains.

V. Let $A \subset \mathbb{Z}/p\mathbb{Z}$, $|A| = n$, where $p > 2^{2^n}$ is a prime. Denote by $s_x(A)$ the number of subsets $S \subseteq A$ such that $\sum_{a \in S} a = x \mod p$. Prove that $s_x(A) = o(2^n)$ for all x.

VI. Let $n = 2k$. A cyclic sequence $(a_1, \ldots, a_N)$, $a_i \in [n]$, is called (n, k)-*universal* if it contains every k-permutation as a subword. Clearly, we must always have $N \ge n!/k!$, and such sequence exists for $N = k(n!/k!)$. Can you find a shorter (n, k)-universal sequence?

VII. In class, we showed that the Ford–Fulkerson algorithm terminates (stops after finitely many steps) for integer capacities. Give an example of a network with capacities in $\mathbb{R}_+$ which does not terminate.

VIII. (a) Suppose every maximal antichain in poset P contains at least 2 elements. Prove that P contains 2 disjoint maximal chains.
(b) Prove or disprove: If every maximal antichain in poset P contains at least 3 elements, then P contains 3 disjoint maximal chains.

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This Homework is due Monday Feb 19, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Make sure you write your **name**, your **UCLA id number**, and your collaborators' names at the front page.

P.S. Each problem above has the same weight.