I. Let $I_i = [a_i, b_i] \subset \mathbb{R}, 1 \leq i \leq n$ be a family of intervals. Denote by $P = ([n], \prec)$ a poset on these intervals with $I_i \prec I_j$ if $b_i < a_j$. These are called I-posets and intervals $\{I_i\}$ are their representations. Prove that $P = (X, \prec)$ is an I-poset if and only if for all $x \prec y$, $u \prec v$ we have $x \prec v$ or $u \prec y$.

II. Prove or disprove:
   a) Every I-poset has a representation with intervals of lengths in $[1, 3]$.
   b) Every I-poset has a representation with intervals of at most 3 distinct lengths.

III. Let $n$ be even. A subset $A \subseteq [n]$ is called nice if the numbers of even and odd elements in $A$ differs by at most 1. Suppose a collection of subsets $\mathcal{A} \subseteq 2^{[n]}$ satisfies the property
   $$A \subset B, \ A, B \in \mathcal{A} \implies B \setminus A \text{ is nice}.$$ 
   Prove that $|\mathcal{A}| \leq \left(\frac{n}{2}\right)$.

IV. A chain in a finite poset $P = (X, \prec)$ is called maximal if it is contained in no other chain. Prove that the maximum number of disjoint maximal chains is equal to the minimum size of a set intersecting all maximal chains.

V. Let $A \subseteq \mathbb{Z}/p\mathbb{Z}, |A| = n$, where $p > 2^n$ is a prime. Denote by $s_x(A)$ the number of subsets $S \subseteq A$ such that $\sum_{a \in S} a \equiv x \pmod{p}$. Prove that $s_x(A) = o(2^n)$ for all $x$.

VI. Let $n = 2k$. A cyclic sequence $(a_1, \ldots, a_N), a_i \in [n]$, is called $(n,k)$-universal if it contains every $k$-permutation as a subword. Clearly, we must always have $N \geq n!/k!$, and such sequence exists for $N = k(n!/k!)$. Can you find a shorter $(n,k)$-universal sequence?

VII. In class, we showed that the Ford–Fulkerson algorithm terminates (stops after finitely many steps) for integer capacities. Give an example of a network with capacities in $\mathbb{R}_+$ which does not terminate.

VIII. (a) Suppose every maximal antichain in poset $P$ contains at least 2 elements. Prove that $P$ contains 2 disjoint maximal chains.
   (b) Prove or disprove: If every maximal antichain in poset $P$ contains at least 3 elements, then $P$ contains 3 disjoint maximal chains.

***************

This Homework is due Monday Feb 19, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Make sure you write your name, your UCLA id number, and your collaborators’ names at the front page.
P.S. Each problem above has the same weight.