HOME ASSIGNMENT 1 (MATH 206B, WINTER 2024)

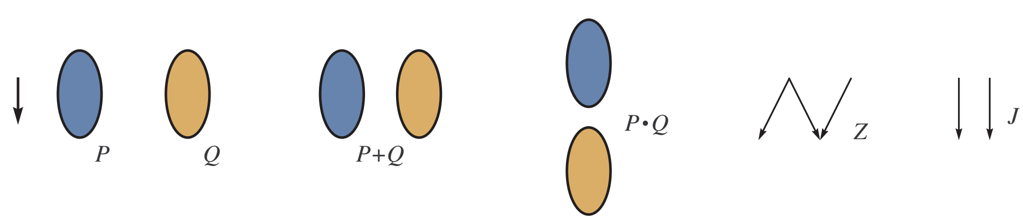
I. Find asymptotics for the number of posets of height two (up to isomorphism). Same for posets of width two. Compare the asymptotics.

II. Let P_n be a poset of subspaces of $\mathbb{F}_2^n$ by inclusion. Compute the number of maximal chains in P_n. Same for the total number of chains in P_n. Use this to give sharp asymptotics.

III. Let n be even. Give asymptotics for the number of antichains in P_n.

IV. Let Q_n be the 2-dim poset of points $\{(i,j), 1 \le i,j \le n\} \subset \mathbb{N}^2$. Compute the number of maximal chains and antichains in Q_n. Give asymptotics for the total number of chains and antichains in Q_n. Compare these asymptotics.

V. For posets $P = (X, \prec)$ and $Q = (Y, \prec')$ define the *sum* $P + Q$ and the *product* $P \cdot Q$ on the same set $X \cup Y$, as in the figure. Define also a 4-element poset Z as in the figure. Prove that a poset A does not contain Z as an *induced subposet* (meaning it has no additional relations), if and only if A can be obtained from a single-element poset using the sum and product operations. We call these posets *nice*



VI. Use Problem V to prove that the number of non-isomorphic posets on n elements which have no induced Z and J as in the figure, is the Catalan number C_n.

VII. Use the Problem V to conclude that testing whether poset $P = (X, \prec)$ on $|X| = n$ elements is *nice* can be done in polynomial time. Can you design a faster algorithm? Say, in quadratic or even nearly linear time?

VIII. Consider all *point-and-line configurations* obtained by taking n (labeled) points in $\mathbb{F}_q^2$ and taking all lines which contain at least 2 of these points. For a poset P of height two with n minimal elements, denote by $f_P(q)$ the number of such configurations isomorphic to P. Prove or disprove: $f_P(q)$ is a polynomial in q for all P, where $q > n^2$ is prime.

IX. Let $P = (X, \prec)$ be a ranked poset with the rank function $\rho : X \to \mathbb{N}$. Denote by $\{n_k\}$ the *rank sequence*: $n_k = |\rho^{-1}(k)|$. We say that P is *well-ranked* if

$$\sum_{x \in A} \frac{1}{n_{\rho(x)}} \leq 1$$

for all antichains A in P. Recall the *LYM theorem* that the *Boolean lattice* $B_n = (2^{[n]}, \subset)$ is well ranked. Prove that well-ranked posets have Sperner property.

X. Prove that the product $P \times Q$ of well-ranked posets P, Q is also well-ranked.

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This Homework is due Monday Feb 5, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Make sure you write your **name**, your **UCLA id number**, and your collaborators' names at the front page.

P.S. Each problem above has the same weight.