I. Find asymptotics for the number of posets of height two (up to isomorphism). Same for posets of width two. Compare the asymptotics.

II. Let $P_n$ be a poset of subspaces of $\mathbb{F}_2^n$ by inclusion. Compute the number of maximal chains in $P_n$. Same for the total number of chains in $P_n$. Use this to give sharp asymptotics.

III. Let $n$ be even. Give asymptotics for the number of antichains in $P_n$.

IV. Let $Q_n$ be the 2-dim poset of points $\{(i, j), 1 \leq i, j \leq n\} \subset \mathbb{N}^2$. Compute the number of maximal chains and antichains in $Q_n$. Give asymptotics for the total number of chains and antichains in $Q_n$. Compare these asymptotics.

V. For posets $P = (X, \prec)$ and $Q = (Y, \prec')$ define the sum $P + Q$ and the product $P \cdot Q$ on the same set $X \cup Y$, as in the figure. Define also a 4-element poset $Z$ as in the figure. Prove that a poset $A$ does not contain $Z$ as an induced subposet (meaning it has no additional relations), if and only if $A$ can be obtained from a single-element poset using the sum and product operations. We call these posets nice.

VI. Use Problem V to prove that the number of non-isomorphic posets on $n$ elements which have no induced $Z$ and $J$ as in the figure, is the Catalan number $C_n$.

VII. Use the Problem V to conclude that testing whether poset $P = (X, \prec)$ on $|X| = n$ elements is nice can be done in polynomial time. Can you design a faster algorithm? Say, in quadratic or even nearly linear time?

VIII. Consider all point-and-line configurations obtained by taking $n$ (labeled) points in $\mathbb{F}_q^2$ and taking all lines which contain at least 2 of these points. For a poset $P$ of height two with $n$ minimal elements, denote by $f_P(q)$ the number of such configurations isomorphic to $P$. Prove or disprove: $f_P(q)$ is a polynomial in $q$ for all $P$, where $q > n^2$ is prime.
IX. Let $P = (X, \preceq)$ be a ranked poset with the rank function $\rho : X \to \mathbb{N}$. Denote by \( \{n_k\} \) the rank sequence: $n_k = |\rho^{-1}(k)|$. We say that $P$ is well-ranked if

$$\sum_{x \in A} \frac{1}{n_{\rho(x)}} \leq 1$$

for all antichains $A$ in $P$. Recall the LYM theorem that the Boolean lattice $B_n = (2^{[n]}, \subset)$ is well ranked. Prove that well-ranked posets have Sperner property.

X. Prove that the product $P \times Q$ of well-ranked posets $P, Q$ is also well-ranked.

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This Homework is due Monday Feb 5, at 1:59 pm (right before class). The solutions must be uploaded to the Gradescope. Please read the collaboration policy on the course web page. Make sure you write your name, your UCLA id number, and your collaborators’ names at the front page.

P.S. Each problem above has the same weight.