HOME ASSIGNMENT 1 (MATH 206B, WINTER 2022)

1) Denote by $q(n)$ the number of partitions of n into distinct parts such that the smallest part is odd. Prove that $q(n)$ is odd if and only if n is a square.

2) Prove the following "large product formula":

$$\prod_{\lambda=(1^{m_1}2^{m_2}\ldots)\vdash n} m_1!\, m_2!\, \cdots \quad = \prod_{\mu=(1^{r_1}2^{r_2}\ldots)\vdash n} 1^{r_1}\, 2^{r_2}\, \cdots$$

3) Prove the following curious identity:

$$\frac{t}{1-t} + \frac{t^3}{1-t^3} + \frac{t^5}{1-t^5} + \frac{t^7}{1-t^7} + \ldots = \frac{t}{1-t} + \frac{t^3}{1-t^2} + \frac{t^6}{1-t^3} + \frac{t^{10}}{1-t^4} + \ldots$$

4) Let A, B be two infinite sets. Suppose there is a one-to-one correspondence between $T \times A$ and $T \times B$, where $T = \{0, 1\}$. Without using the *Axiom of Choice*, prove that there is a one-to-one correspondence between A and B.

5) Prove directly that the *theta function* $\theta(q) = \sum_{m=-\infty}^{\infty} q^{m^2}$ satisfies an *algebraic differential equation* (ADE): $F(\theta, \theta', \ldots, \theta^{(k)}, q) = 0$ for some polynomial $F \in \mathbb{Z}[x_0, x_1, \ldots, x_{k+1}]$. You can use Jacobi's two, four, six and eight squares theorems and Jacobi's triple product identity, but nothing from the 20th or 21st Century.

6) Let $\alpha(n)$ be the number of $1 \le k \le n$ such that $p(k)$ is odd. Prove that $\alpha(n) = \Omega(\sqrt{n})$.

7) Denote by $a(n)$ the number of partitions of n into parts $1, 2, 4, 8, \ldots$ For example, $a(4) = 4$ since $4 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. Prove that $a(n) = n^{\Theta(\log n)}$.

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This Homework is due Wednesday Feb. 16, at 2:59 pm (right before class). Please upload your solution to the Gradescope. Please read the collaboration policy on the course website. Make sure you write your name in the beginning and your collaborators' names at the end.

P.S. Each item above has the same weight.