

Algebraic Combinatorics (Math 206B, Winter 2019)

Instructor: Igor Pak

pak@math.ucla (add .edu at the end)

Class schedule: MWF 1:00 - 1:50 pm, MS 7608

Office Hours: M 2-3, MS 6125

Textbooks:

- R.P. Stanley, *Enumerative Combinatorics*, vol. 2.
-

Notes:

See notes by [Daniel Raban](#) available [here](#). These are unedited and unverified.

Click [here](#) to return to Igor Pak Home Page.

Last updated 4/26/2021.

Math 206B Lecture 1 Notes

Daniel Raban

January 7, 2019

1 Teaser of Course Topics

This lecture will be an advertisement for the topics in the course.¹

1.1 Combinatorics of S_n and applications

We denote $[n] := \{1, \dots, n\}$.

Definition 1.1. The **symmetric group** S_n is the group of bijections $\sigma : [n] \rightarrow [n]$.

We can write permutations as products of cycles.

Example 1.1. The permutation $\sigma = (5 \ 1 \ 2 \ 4 \ 3)$ represents the bijection sending $1 \mapsto 5$, $2 \mapsto 1$, etc.

The conjugacy classes of S_n are the different cycle types. These correspond to partitions of n . Let $p(n)$ be the number of conjugacy classes of S_n . Euler showed that

$$1 + \sum_{n=1}^{\infty} p(n)t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}.$$

What does this all have to do with the symmetric group itself? Here is Percy MacMahon's version of the story. If we have a partition, we can think of it as a sequence of numbers, padded with zeros at the end to make it infinite. We can write

$$\sum_{\lambda \in \mathcal{P}} t^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1}{1-t^i}.$$

If we write the partition over and over in a grid (chopping off an element from the front each time, we can get what is called a **plane partition**. This gives us

$$\sum_{A \in \mathcal{PP}} t^{|A|} = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)^i},$$

¹This is an advertisement of an advertisement, much like the ad before watching a trailer online.

where $|A| = \sum_{i,j} a_{i,j}$ is the sum of the numbers in the plane partition. Why should this be true? This is actually related to the representation theory of S_n .

MacMahon made a conjecture for higher dimensions:

Theorem 1.1.

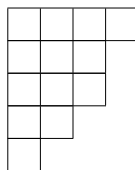
$$\sum_{A \in \mathcal{P}(d)} t^{|A|} = \prod \frac{1}{(1 - t^i)^{(i_{d-1})}}.$$

This does not work. It actually fails for $d = 3$ and a low coefficient like the coefficient of t^7 . MacMahon did not understand why the formula was true, even though he proved it. Irreducible representations of S_n will correspond to partitions of n $\mathcal{P}_n := \{\lambda \in \mathcal{P} : |\lambda| = n\}$ because these correspond to conjugacy classes of S_n .

Theorem 1.2 (A. Young, 1897). *Let f^λ be the dimension of S^λ . Then f^λ is the number of standard Young tableau with shape λ .*

Definition 1.2. Given a partition λ , the *young diagram* of λ is the partition expressed as stacked rows of boxes.

Example 1.2. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 1.3. A **Young tableau** is a Young diagram where we fill in the boxes with the numbers 1 to n , according to the rule that the numbers have to be increasing going to the right and going down.

Example 1.3. Here is a Young tableau:

1	2	3	7
4	5	10	
6	8		
9	12		
11			

Theorem 1.3 (FRT, c.1960). $f^\lambda = \frac{n!}{\prod_{i,j} h_{i,j}}$, where $h_{i,j}$ is the length of the hook starting from position i, j and going to the right and downwards.

From representation theory, we can get that $f^\lambda \mid |S_n|$, so we know that f^λ is $n!$ divided by something. The magic is in what that something is.

1.2 Representation theory of $\mathrm{GL}_n(\mathbb{C})$

A basic representation of $\mathrm{GL}_n(\mathbb{C})$ is ρ_M is the matrix M acting on \mathbb{C}^n . Another representation is the determinant map. We will find that representations of $\mathrm{GL}_n(\mathbb{C})$ correspond to sequences $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

Theorem 1.4 (Weyl). $\dim(\rho_\lambda)$ equals the number of semistandard Young tableau with shape λ .

This will be another remarkable product formula. This, paired with the hook-length formula, will pave the way for a nice proof of MacMahon's formula.

1.3 Young graph

Definition 1.4. The **Young graph** is the graph with vertices $\lambda \in \Gamma$, and edges (λ, μ) where $\mu \setminus \lambda$ is a single difference.

Essentially, we have taken all young diagrams and made an undirected graph, partially ordering them by containment.

Theorem 1.5. The number of loops of length $2n$ (that do not zigzag up and down) in the Young graph is $n!$.

Proof. We can prove this using basic representation theory.

$$\# \text{ loops} = \sum_{\lambda \in \mathcal{P}_n} (\# \text{ paths } \phi \rightarrow \lambda)^2 = \sum_{\lambda \in \mathcal{P}_n} \mathrm{SYT}(\lambda)^2 = \sum_{\lambda \in \mathcal{P}_n} (f^\lambda)^2 = |S_n| = n! \quad \square$$

What about general loops?

Theorem 1.6. The number of general loops is $(2n - 1)!!$.

Math 206B Lecture 2 Notes

Daniel Raban

January 9, 2019

1 Representation Theory of Finite Groups

This is meant to remind you of basic results in representation theory. Good references are Chapter 1 of Sagan's book and the book by Fulton and Harris.¹

1.1 Conjugacy classes and characters

Let G be a finite group. There is an action of G on G in the following way: $a \cdot g = aga^{-1}$. This is the action of conjugation, where conjugacy classes are the orbits of the action. In other words, $g \sim h$ if $g = aha^{-1}$ for some $a \in G$; conjugacy classes are equivalence classes under this relation. We will denote $c(G)$ as the number of conjugacy classes of G .

Example 1.1. Let $G = \mathbb{Z}_n$ be the cyclic group of order n . Then $c(\mathbb{Z}_n) = n$.

Example 1.2. Let $G = S_n$. Then $c(S_n) = p(n)$, the number of integer partitions of n .

Definition 1.1. A **character** $\chi : G \rightarrow \mathbb{C}$ is a function such that $\chi(g) = \chi(h)$ whenever $g \sim h$.

Example 1.3. The trivial character is $\chi(g) = 1$ for all g .

Example 1.4. Let $G = S_n$. The sign character is $\chi(\sigma) = \text{sign}(\sigma)$.

Theorem 1.1. $\dim(\text{span}(\text{characters } \chi)) = c(G)$.

Definition 1.2. The **inner product** on characters is defined as

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Example 1.5. If $n \geq 2$, then

$$\langle \chi, \text{sign} \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot 1 = \frac{1}{n!} \left(\frac{n!}{2} - \frac{n!}{2} \right) = 0.$$

Example 1.6. Suppose $G = \mathbb{Z}_n$. Let $\omega = e^{2\pi i/n}$. For each j , we have a character $\chi_j(k) = \omega^{jk}$.

¹This is Professor Pak's opinion. I hate this book.

1.2 Linear representations

Let $V = \mathbb{C}^d$. The group $\mathrm{GL}(V) = \mathrm{GL}_d(\mathbb{C})$ is the group of automorphisms of V .

Definition 1.3. A **representation** is a group homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$.

Here are operations we can do on representations:

1. If we have $\rho : G \rightarrow \mathrm{GL}(V), \pi : G \rightarrow \mathrm{GL}(W)$, then we can form $\rho \oplus \pi : G \rightarrow \mathrm{GL}(V \oplus W)$ by acting on the individual parts of the vector space by the respective actions.
2. We can form $\rho \otimes \pi : G \rightarrow \mathrm{GL}(V \otimes W)$. The dimension of $\rho \otimes \pi$ is $\dim(V) \dim(W)$.
3. Reduced representations: If we have $\rho : G \rightarrow \mathrm{GL}(V)$ and $H \leq G$, we can define $\rho \downarrow_H^G$ as the restriction of ρ to H .
4. Induced representations: If we have $\pi : H \rightarrow \mathrm{GL}(W)$, there is an induced representation $\pi \uparrow_H^G : G \rightarrow \mathrm{GL}(W^{\otimes h})$, where $h := [G : H] = |G|/|H|$.

Example 1.7. The **trivial representation** maps $g \mapsto \mathrm{id}_V$ for all $g \in G$.

Example 1.8. The **regular representation** $\pi : G \rightarrow \mathrm{GL}(\mathbb{C}^{|G|})$ acts on a basis indexed by all $g \in G$ by $a \cdot v_g = v_{ag}$.

Example 1.9. The **natural representation** $\rho : S_n \rightarrow \mathrm{GL}_n(\mathbb{C})$ sends σ to its permutation matrix (applying the permutation to the basis vectors $\{e_1, \dots, e_n\}$).

If we have a representation $\rho : G \rightarrow \mathrm{GL}(V)$, we can define its character χ_ρ by $\chi_\rho(g) = \mathrm{tr}(\rho(g))$. Since $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, $\mathrm{tr}(BAB^{-1}) = \mathrm{tr}(A)$. So χ_ρ is in fact a character in the previous sense.

Remark 1.1. Even though $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, then $\mathrm{tr}(ABC) \neq \mathrm{tr}(CBA)$.

We also have $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. However, we do not have $\mathrm{tr}(AB) = \mathrm{tr}(A) \mathrm{tr}(B)$.

Example 1.10. Let π be the regular representation of G . Then

$$\chi_\pi(\sigma) = \begin{cases} n! & \sigma = 1 \\ 0 & \sigma \neq 1. \end{cases}$$

Example 1.11. Let ρ be the natural representation of S_n . Then $\chi_\rho(\sigma)$ is the number of fixed points of σ .

Example 1.12. Here is an example of a reduced representation. Let $G = S_n$, and $H = \mathbb{Z}_n$. Let ρ be the natural representation of S_n . Then $\rho \downarrow_{\mathbb{Z}_n}^{S_n}$ is the regular representation of \mathbb{Z}_n .

Math 206B Lecture 3 Notes

Daniel Raban

January 11, 2019

1 Irreducible Representations and Characters

1.1 Irreducible representations

Recall that if π, ρ are representations of G , then $\pi \oplus \rho$ is well-defined.

Definition 1.1. A representation ρ is **irreducible** if $\rho \neq \pi \oplus \pi'$.

Theorem 1.1. *Let G be a finite group. There exists a finite number of irreducible representations ρ_0, ρ_1, \dots such that every $\pi = \bigoplus m_i \rho_i$ for some coefficients m_i . Moreover, $m_i = \langle \chi_\pi, \chi_{\rho_i} \rangle$.*

In other words, $\{\chi_{\rho_i}\}$ is an orthonormal basis in the space of functions $f : G \rightarrow \mathbb{C}$ that are constant on conjugacy classes.

Remark 1.1. The decomposition $\pi = \bigoplus m_i \rho_i$ is non-unique. G can act on $V = \mathbb{C}^d$ trivially. This representation is $\pi = \bigoplus d \rho_0$. Any basis will work. When all m_i are 0 or 1, this decomposition is in fact unique.

Example 1.1. Let π be the regular representation of G . Let χ_i be the character of an irreducible representation. Then $m_i = \langle \chi_\pi, \chi_i \rangle = |G|^{-1} \sum_{g \in G} \chi_\pi(g) \overline{\chi_i(g)} = |G|^{-1} \chi_\pi(1) \chi_i(1) = |G|^{-1} |G| \dim(\rho_i)$.

1.2 Irreducible characters and character tables

Let $\chi_0, \dots, \chi_{c-1}$ be the characters of the irreducible representations of G , where $c = c(G)$. Then $\langle \chi_i, \chi_j \rangle = \delta_{i,j}$. Let $d_i = \chi_i(1) = \dim(\rho_i)$.

Theorem 1.2. $\sum_i d_i^2 = |G|$.

Theorem 1.3. $d_i \mid |G|$ for all i .

Theorem 1.4. All χ_i are real iff $C^{-1} = C$ for all conjugacy classes C of G .

Let's calculate character tables of irreducible representations of S_n .

Example 1.2.

S_2	id	(1 2)
χ_0	1	1
χ_1	1	-1

For every n , there is the character $\chi_{\text{sign}}(\sigma) = (-1)^{\text{inv}(\sigma)}$, where $\text{inv}(\sigma)$ is the number of the inversions.

Proposition 1.1. $\text{inv}(\sigma\tau) \cong \text{inv}(\sigma) + \text{inv}(\tau) \pmod{2}$.

Here is the character table for S_3 :

Example 1.3.

S_3	id	(1 2)	(1 2 3)
χ_0	1	1	1
χ_{sign}	1	-1	1
χ_1	2	0	-1

How do we find the values for χ_1 ? The first value is the dimension of the representation. Since $\chi_0 + \chi_{\text{sign}} + 2\chi_1 = \chi_\pi$, we can figure out the rest of the values.

What is the representation corresponding to the character χ_1 ? We can calculate $\langle \chi_0, \chi_{\text{nat}} \rangle = \dim(\rho^G)$ or, combinatorially, $= |G|^{-1} \sum_{\sigma \in S_n} \text{fixed pts}(\sigma) \cdot 1 = \frac{1}{n!} \sum_{i=1}^n \sum_{\sigma(i)=i} 1 = \frac{1}{n!} n(n-1)! = 1$. Then $\chi_1 = \chi_{\text{nat}} - \chi_0$.

Example 1.4.

S_4	id	(1 2)	(1 2 3)	(1 2)(3 4)	(1 2 3 4)
χ_0	1	1	1	1	1
χ_{sign}	1	-1	1	1	-1
χ_1	3	1	0	-1	-1
χ'_1	3	-1	0	-1	1
χ_2	2	0	-1	2	0

We can figure out χ_1 as the number of fixed points minus 1. Here, $\chi'_1 = \chi_1 \cdot \chi_{\text{sign}}$. We can figure out χ_2 using the regular representation, as before.

Math 206B Lecture 4 Notes

Daniel Raban

January 14, 2019

1 Characters of S_n

1.1 Induced M^μ representations and μ -flags

Last time, we found a character of S_4 , but we didn't quite know what representation it corresponded to. Let's try to understand this a little better.

Definition 1.1. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n . Define

$$M^\mu = \text{ind}_{S_{\mu_1} \times \dots \times S_{\mu_k}}^{S_n} 1.$$

Example 1.1. Let $\mu = (n-1, 1)$. then $M^{n-1,1} = \text{ind}_{S_{n-1}}^{S_n} 1$. $\dim(M^{n-1,1}) = n$. This is the natural representation.

Definition 1.2. Let $\mu = (\mu_1, \dots, \mu_k)$ be a partition of n . A μ -flag on $[n]$ is $\emptyset \subseteq A_1 \subseteq A_2 \subseteq \dots \subseteq [n]$ such that $|A_1| = \mu_1$, $|A_2| = \mu_1 + \mu_2$, and so on.

Example 1.2. Let $\mu = (n-k, k)$, where $1 \leq k \leq n/2$. Then μ -flags are in correspondence with $(n-k)$ subsets of $[n]$: $\emptyset \subseteq A_1 \subseteq [n]$.

Example 1.3. Let $\mu = (1^n)$. μ -flags are in correspondence with S_n , where $\sigma \mapsto A_1 \subseteq \dots \subseteq A_n$, and $A_i = \{\sigma(1), \sigma(2), \dots, \sigma(i)\}$.

1.2 Structure of the M^μ representations of S_n

Definition 1.3. Let G be a finite group, and let X be a finite set. Let $G \curvearrowright X$. Then there is a **permutation representation** $\varphi : G \rightarrow S_X$.

Equivalently, there is a permutation representation $\rho_\varphi : G \rightarrow \text{GL}(V)$ over \mathbb{C} , where $V = \mathbb{C}\langle x \rangle$.

Proposition 1.1. M^μ is a permutation representation of S_n on μ -flags of $[n] = \{1, \dots, n\}$.

Example 1.4. Let $\mu = (2^2)$. Then M^μ is the permutation representation of S_n on 2-subsets of $[4]$. $\dim(M^{2^2}) = \binom{4}{2} = 6$. We claim that $M^{2^2} = S^{(4)} \oplus S^{(3,1)} \oplus S^{(2^2)}$, where these refer to the irreducible representations of S_4 . Let's calculate $\chi_{M^{2^2}}$:

λ	1^4	$2\ 1^2$	2^2	$3\ 1$	4
$\chi_{M^{2^2}}$	6	2	2	0	0
z_λ	1	6	3	8	6

Proposition 1.2. Let $\nu = (\nu_1, \nu_2, \dots) = 1^{m_1(\nu)} 2^{m_2(\nu)} \dots$ be a partition of n , where m_i is the number of i s in ν . Then

$$z_\nu = \frac{n!}{(m_1! 1^{m_1})(m_2! 2^{m_2}) \dots}.$$

Theorem 1.1. $M^\mu = \bigoplus_\lambda m_{\mu,\lambda} S^\lambda$, where

$$m_{\mu,\lambda} := \langle M^\mu, S^\lambda \rangle = \frac{1}{n!} \sum_\nu z_\nu \chi_{M^\mu}[\nu] \chi_{S^\lambda}[\nu].$$

Theorem 1.2. The matrix $[m_{\mu,\lambda}]$ has nonnegative integer entries and is upper triangular with 1s on the diagonal, where $\lambda \leq \mu$ is the lexicographic order.

Math 206B Lecture 5 Notes

Daniel Raban

January 16, 2019

1 Tableau

1.1 Young diagrams and tableau

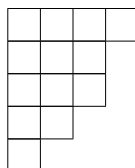
We want to work toward the following theorem:

Theorem 1.1. $M^\mu = \bigoplus_{\lambda \geq \mu} m_{\mu, \lambda} S^\lambda$, where

$$m_{\mu, \lambda} := \langle M^\mu, S^\lambda \rangle = \frac{1}{n!} \sum_{\nu} z_{\nu} \chi_{M^\mu}[\nu] \chi_{S^\lambda}[\nu].$$

Definition 1.1. Given a partition λ , the **Young diagram** of λ is the partition expressed as stacked rows of boxes.

Example 1.1. Take $\lambda = (4, 3, 3, 2, 1)$. The Young diagram of λ is



Definition 1.2. A **standard Young tableau**¹ is a Young diagram where we fill in the boxes with the numbers 1 to n , according to the rule that the numbers have to be increasing going to the right and going down.

Example 1.2. Here is a Young tableau:

1	2	3	7
4	5	10	
6	8		
9	12		
11			

¹The plural of tableau is tableaux.

1.2 Poly-tabloids and their relation to irreducible representations of S_n

Definition 1.3. A **tabloid** is a Young diagram filled with the numbers $\{1, \dots, n\}$. A **standard tabloid** is the same thing, except you need that numbers are increasing going to the right.

Example 1.3. Here is a tabloid:

3	5	6	8	10	11
1	2	7	13		
4	12				

Observe that λ -tabloids are in bijection with λ -flags.

Definition 1.4. A **poly-tabloid** is an equivalence class of tabloids under the action of $S_{\lambda_1} \times S_{\lambda_2} \times \dots$.

Let $R^\lambda = S_{\lambda_1} \times S_{\lambda_2} \times \dots$ be the group of row permutations on λ -tabloids, and let $C^\lambda = S_{\lambda'_1} \times S_{\lambda'_2} \times \dots$ be the group of column permutations on λ -tabloids.

Let

$$\mathcal{X}_\lambda = \sum_{\sigma \in C^\lambda} \text{sign}(\sigma) \sigma$$

be an element of the group algebra $\mathbb{C}[S_n]$.

Example 1.4. Let $\lambda = (3 \ 2)$.

1	2	3
4	5	

Then $\mathcal{X}_\lambda = (1 - (1 \ 4))(1 - (2 \ 5))$, where 1 is the identity in the group algebra.

We can also think of M^λ as S_n acting on $\mathbb{C} \langle \lambda - \text{poly-tabloids} \rangle$. Define $e_t := \mathcal{X}_t \{t\}$, where \mathcal{X}_t is the projection of \mathcal{X} onto $\mathbb{C} \langle \{t\} \rangle$.

Example 1.5. Suppose

$$t = \begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \hline \end{array}$$

Then

$$e_t = \begin{array}{|c|c|c|} \hline 4 & 1 & 2 \\ \hline 3 & 5 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline 4 & 5 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline 4 & 5 & 2 \\ \hline 3 & 1 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline 3 & 5 & 2 \\ \hline 4 & 5 & \\ \hline \end{array}$$

Theorem 1.2. The S_n action on e_t for t a λ -tabloid is an irreducible representation S^λ .

What is the idea here? Start with λ , and construct a λ -tabloid t . Now \mathcal{X}_t is the element of $\mathbb{C}[S_n]$ corresponding to t . Then $e_t = \mathcal{X}_t \{t\}$ is a linear combination of poly-tabloids. If we let $W_\lambda = \mathbb{C} \langle \{t\} \rangle$ be the linear span of poly-tabloids (M^λ acts on W_λ), then $\sigma \cdot e_t \in E_\lambda$. Now we can think of $\mathbb{C} \langle \{\sigma e_t : \sigma \in S_n\} \rangle$. The claim is that this is isomorphic to S^λ . That is, if we define S^λ like this, then the claim is that these are all irreducible and distinct.

Math 206B Lecture 6 Notes

Daniel Raban

January 18, 2019

1 Examples, Dimension, and Irreducibility of S^λ

1.1 Examples of S^λ

Let t be a tabloid, and consider $\mathbb{C} \langle \sigma \cdot e_t : \sigma \in S_n \rangle = S^\lambda$.

Example 1.1. Let $\lambda = (n)$. There is a single poly-tabloid,

1	2	3	\cdots	n
---	---	---	----------	-----

S_n acts trivially on this, so $S^{(n)} = 1$.

Example 1.2. Let $\lambda = (1^n)$. The poly-tabloids correspond to permutations:

$\sigma \mapsto$	$\sigma(1)$
	$\sigma(2)$
	\vdots
	$\sigma(n)$

Let $\sum_{\sigma \in S_n} \text{sign}(\sigma)\sigma = \mathcal{X} \in \mathbb{C}[S_n]$. Then $\pi \cdot \mathcal{X} = \text{sign}(\pi)\mathcal{X}$.

Example 1.3. Let $\nu_k = (n - k, 1^k)$. Then $M^{\nu_k} = \text{ind}_{S_{n-k} \times 1 \times \cdots \times 1}^{S_n} 1$. The dimension is $\dim(M^{\nu_k}) = n!/(n - k)!$. What is S^{ν_k} ? Permutations do not act on the columns, but they permute the elements in the first column. Let $v = \sum_{\sigma \in S_{k+1}} \text{sign}(\sigma)\sigma$. What is $\pi \cdot v = w$? In general $\mathbb{C} \langle \pi e_t \rangle$ will be a representation of S_n .

Example 1.4. Let $\lambda = (2, 2)$. Let

$t =$	1	2
	3	4

Then

$$v = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 4 & 2 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Then $(1\ 2)v = v$ and $(3\ 4)v = v$. We can also calculate

$$(3\ 4)v = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{|c|c|} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array} := w.$$

Then M^μ is a 2-dimensional vector space spanned by v, w .

1.2 Dimension and irreducibility of S^λ

Lemma 1.1. *Let S^λ be as above.*

1. $\dim(S^\lambda) = f^\lambda = \#\text{SYT}(\lambda)$.
2. $n! \leq \sum_\lambda (f^\lambda)^2$, where equality holds iff S^λ is irreducible for each λ .

Definition 1.1. The **dominance order** on partitions is the partial order $\lambda \leq \mu$ if $\lambda_1 + \dots + \lambda_k \geq \mu_1 + \dots + \mu_k$ for all k .

We had that $M^\lambda = \bigoplus_{\mu \leq \lambda} m_{\lambda, \mu} S^\mu$. The proof of the inequality in the 2nd statement comes from this fact.

How should we prove the equality in statement 2 of the lemma? Suppose G is a finite group. We have both a left and a right action of $G \curvearrowright \mathbb{C}[G]$. This gives us a $G \times G$ representation, $\bigoplus_\pi \pi \otimes \pi$, where the sum is over all irreducible representations. We want to view this equation as $n!$ being the order of the group, and the $(f^\lambda)^2$ terms being the tensor product of two irreducible representations. It is hard to do this in general, but it will work in the symmetric group. This is called the RSK correspondence.

Math 206B Lecture 7 Notes

Daniel Raban

January 23, 2019

1 The Robinson-Schensted Algorithm

1.1 Description of the algorithm

Recall Burnside's identity:

Proposition 1.1.

$$\sum_{\lambda} (f^{\lambda})^2 = n!,$$

where

$$f^{\lambda} = \dim(S^{\lambda}) = \chi^{\lambda}(1) = \#\text{SYT}(\lambda).$$

We want a bijection between the symmetric group and the set of pairs (A, B) , where A, B are $\text{SYT}(\lambda)$ for some partition λ of n . We usually write the bijection Φ as $\sigma \mapsto (P, Q)$, where P is called the **insertion tableau** and Q is called the **recording tableau**.

Example 1.1. Let $\sigma = 4\ 2\ 7\ 3\ 6\ 1\ 5$ be a permutation. For each partial reading of the string representing σ , the algorithm will produce an outcome.

First, we start with the 4. Here are our two tableaux.

4	1
---	---

Now let's add the 2. It can't go to the right of the 4, so it pushes the 4 down. We record the move into our right (recording) tableau:

2	1
4	2

Now we add the 7. It is bigger than the 2, so it goes to the right.

2	7	1	3
4		2	

Now we add the 3. It bumps down the smallest number in the first row that is bigger than it, the 7. When the 7 moves down, it does not need to bump the 4:

2	3
4	7

1	3
2	4

Now let's add in the 6.

2	3	6
4	7	

1	3	5
2	4	

What happens when we put in the 1? It is smaller than 2, so it bumps the 2 down. Doing the same process, the 2 is going to bump down the 4:

1	3	6
2	7	
4		

1	3	5
2	4	
6		

Finally, we can add in the 5:

$$P = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 & \\ \hline 4 & 7 & \\ \hline \end{array}$$

$$Q = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline 6 & 7 & \\ \hline \end{array}$$

Theorem 1.1 (R-S, 1961). *The map $\Phi : S_n \rightarrow \{(A, B) : A, B \in \text{SYT}(\lambda), |\lambda| = n\}$ is a bijection.*

Proof. Here are the steps:

1. The map Φ is well defined.
2. $P, Q \in \text{SYT}(\lambda)$ for some partition λ of n .
3. Φ^{-1} is well-defined.

The first two are easy to convince yourself of. For the third, run the algorithm in reverse and see that it outputs the original permutation.¹ \square

¹Back in the 60s, people used to listen to songs backwards to find hidden messages. This is same same, except there is actually a message if you go backwards.

1.2 Properties of the RS algorithm

This bijection is natural in some sense. It actually exhibits the following remarkable property!

Theorem 1.2. *Suppose $\Phi(\sigma) = (P, Q)$. Then $\Phi(\sigma^{-1}) = (Q, P)$.*

We will not prove this now, but we will prove it later in the course.

Theorem 1.3 (Schensted). *Let $\Phi(\sigma) = (P, Q)$, where $P, Q \in \text{SYT}(\lambda)$. Then λ_1 is the length of the longest increasing subsequence in σ .*

Proof. Proceed by induction. We claim that λ_1 of P_i is the longest increasing subsequence of $\sigma = \sigma_1 \cdots \sigma_i$, where P_i is the insertion tableau at the i -th step. Suppose we have $[a_1 \cdots a_r]\sigma_{i+1}$, where $a_1 \cdots a_r$ is the longest increasing subsequence in the permutation. If $\sigma_{i+1} > a_r$, then we must add a number to the first row because nothing gets bumped down. If $a_r > \sigma_{i+1}$, then something in the first row gets bumped down. \square

How does one come up with an algorithm like this? Schensted was a graduate student at either Berkeley or Stanford.² Schensted had a roommate, Floyd, who later became a famous computer scientist. They were interested in sorting things like solitaire (place the newest card on the heap with the smallest number). Schensted saw the above property and found it interesting. The rest of the story will have to wait until next time.

²Professor Pak doesn't remember which.

Math 206B Lecture 9 Notes

Daniel Raban

January 28, 2019

1 The RSK Algorithm

1.1 Semi-standard Young tableaux

We will extend the RS-algorithm, which gave a bijection $\Phi : S_n \rightarrow \coprod_{|\lambda|=n} \text{SYT}(\lambda)^2$. RSK will give a bijection $\Phi : M(\bar{a}, \bar{b}) \rightarrow \coprod_{|\lambda|=N} \text{SSYT}(\lambda, \bar{a}) \times \text{SSYT}(\lambda, \bar{b})$. Here, $\bar{a} = (a_1, \dots, a_n)$, $\bar{b} = (b_1, \dots, b_n)$, and $N = |\bar{a}| = |\bar{b}| = a_1 + \dots + a_n = b_1 + \dots + b_n$. $M(\bar{a}, \bar{b})$ is the set of \mathbb{N}^+ $n \times n$ matrices with row sums a_1, \dots, a_n and column sums b_1, \dots, b_n . $\text{SSYT}(\lambda, \bar{a})$ is the set of semi-standard Young tableaux of shape λ and weight \bar{a} .

Definition 1.1. A **semi-standard Young tableau** of shape λ is a Young tableau where we are allowed to have numbers reused, and we have numbers are weakly increasing as we go to the right. The **weight** of a semi-standard Young tableau is (m_1, m_2, \dots) , where m_i is the number of i s in A .

Example 1.1. If $\bar{a} = \bar{e} = (1, \dots, 1)$, then $\text{SSYT}(\lambda, \bar{a}) = \text{SYT}(\lambda)$. Then $M(\bar{a}, \mathbf{a})$ is the number of 0-1 matrices with row sums equal to 1 and column sums equal to 1. So the number of such matrices is $|S_n| = n!$. This special case is the case of R-S.

Example 1.2. Let $n = 2$, and $\bar{a} = (m, m) = \bar{b}$. Then $\#M(\bar{a}, \bar{b}) = m + 1$ because the entire matrix is determined by the upper left entry:

$$\begin{bmatrix} * & \\ & \end{bmatrix}$$

The right hand side is $\coprod_{|\lambda|=2m} \text{SSYT}(\lambda, \bar{a})^2$. Note that $\lambda = (\lambda_1, \lambda_2)$. Otherwise, $\text{SSYT}(\lambda, \bar{a}) = 0$. The number of such λ is $m + 1$:

1	1	...	1	1	1
2	...	2			

This case is very different from the R-S case.

1.2 Description of the algorithm

Given $M \in M(\bar{a}, \bar{b})$, we first need to turn M into a word.

Example 1.3. Here is how we turn a matrix into a word.

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 1 \\ 3 & 1 & 1 \end{bmatrix} \rightarrow 1 \ 1 \ 3 \ 3 \ 3 \ 1 \ 2 \ 2 \ 2 \ 2 \ 3 \ 1 \ 1 \ 1 \ 2 \ 3$$

We have 2 1s, then 0 2s, then 3 3s. Then we have 1 1, 4 2s, and 1 3. Continue like this.

Now we will proceed by applying the R-S bumping procedure to this word.

Example 1.4. Start with the previous word. Our partial outputs are

1	1	3	3	3
---	---	---	---	---

1	1	1	3	3
3				

1	1	1	2	3
3	3			

1	1	1	2	2
3	3	3		

and so on.

This gives us an $A \in \text{SSYT}(\lambda, \bar{b})$. How do we make our corresponding recording tableau? The numbers that come from row i on the matrix get recorded in our recording tableau as i . We fill in the shape of the insertion tableau as the shape of the insertion tableau evolves.

Example 1.5. Our partial outputs for the recording tableau are

1	1	1	1	1
---	---	---	---	---

1	1	1	3	3
2				

1	1	1	2	2
2	2			

1	1	1	1	1
2	2	2		

and so on.

1.3 Properties and relationship to representation theory

Theorem 1.1 (Knuth, c. 1980). *RSK is a bijection.*

Proof. Here is what we need to show:

1. Φ is well-defined.
2. Φ^{-1} is well-defined.

Like before, we can prove these step by step and induct on the number of steps. \square

Theorem 1.2. *If $\Phi(M) = (A, B)$, then $\Phi(M^\top) = (B, A)$.*

From representation theory, we had the decomposition $M^\mu = \bigoplus_{|\lambda|=n} m_{\lambda,\mu} S^\lambda$. Here, $M^\mu = \text{ind}_{S_{\mu_1} \times S_{\mu_2} \times \dots}^{S_n} 1$. Take $\bar{a} = (a_1, \dots, a_k)$. We can define $M^{\bar{a}} = \text{ind}_{S_{a_1} \times S_{a_2} \times \dots}^{S_N} 1$, where $N = a_1 + \dots + a_k$.

Theorem 1.3.

$$M^{\bar{a}} = \bigoplus_{|\lambda|=N} m_{\lambda,\bar{a}} S^\lambda,$$

where $m_{\lambda,\bar{a}} = \# \text{SSYT}(\lambda, \bar{a})$.

Why should this be true?

$$\#M(\bar{a}, 1^N) = \frac{N!}{a_1! a_2! \dots} = \dim(M^{\bar{a}}).$$

$$M(\bar{a}, 1^N) = \sum_{|\lambda|=N} \# \in \text{SSYT}(\lambda, \mu) \cdot \# \text{SYT}(\lambda).$$

Now

$$M(\bar{a}, \bar{b}) = \langle \chi_{M^{\bar{a}}}, \chi_{M^{\bar{b}}} \rangle = \dim \text{Hom}(M^{\bar{a}}, M^{\bar{b}}).$$

Math 206B Lecture 10 Notes

Daniel Raban

January 30, 2019

1 Geometric RSK

1.1 Longest value of a path in RSK

Let Φ denote the R-S algorithm, and let $\hat{\Phi}$ denote RSK. For Φ , we had a property about the longest increasing subsequence of a permutation. What about for $\hat{\Phi}$?

Example 1.1. When we run RSK on

$$\begin{bmatrix} 2 & 0 & 3 \\ 1 & 4 & 1 \\ 5 & 3 & 1 \end{bmatrix}$$

we get the pair of tableau

1	1	1	1	1	1	1	1	1	2	2	2	3
2	2	2	2	3								
3	3	3										

1	1	1	1	1	2	2	2	3	3	3	3
2	2	2	3	3							
3	3	3									

Then $\lambda_1 = 12$, which happens to be the sum of the numbers in the longest path from top left to bottom right : $2 + 1 + 5 + 3 + 1 = 12$.

Theorem 1.1. *Let $\hat{\Phi}(M) = (A, B)$ have shape λ . Then $\lambda = \gamma(M)$, where γ is the maximum total value of a path from $(1, 1)$ to (n, n) .*

1.2 Geometric RSK

How would we feed a semistandard Young tableau to a computer? We want to think of $A = (\lambda \supseteq \cdots \supseteq \mu^{(2)} \supseteq \mu^{(1)})$, where μ_i is the shape of the tableau, only looking at the numbers $\leq i$.

Example 1.2. For the tableaux

1	1	1	1	1	1	1	1	2	2	2	3
2	2	2	2	3							
3	3	3									

1	1	1	1	1	2	2	2	3	3	3	3
2	2	2	3	3							
3	3	3									

we get

$$\begin{array}{ccccc} 12 & 11 & 8 & & 12 \\ & 5 & 4 & & 8 & 5 \\ & & 3 & & 5 & 3 & 3 \end{array}$$

We can put these together into a matrix

$$\begin{bmatrix} 12 & 11 & 8 \\ 8 & 5 & 4 \\ 5 & 3 & 3 \end{bmatrix}.$$

The output of RSK can then be sent to a computer as $x = (x_{i,j})$ such that

- $x_{i,j} \geq 0$
- $x_{i,j} \leq x_{i,j+1}, x_{i+1,j}$
- $\sum_{i-j=n-c} x_{i,j} = a_1 + \cdots + a_c$
- $\sum_{j-i=n-c} x_{i,j} = b_1 + \cdots + b_c$, where $0 \leq c \leq n$.

This defines a polytope. So we can think of $\tilde{\Phi} : \tilde{M}(\bar{a}, \bar{b}) \rightarrow \tilde{X}(a, b)$, where the left hand side takes a polytope defined by a matrix, and the right side outputs a polytope defined by a matrix.

Theorem 1.2. $\tilde{\Phi}$ is piecewise linear, volume preserving, and continuous.

These polytopes were invented by Gelfand and Tseitin. This was further developed in a paper by Gelfand and Zelevinsky.

1.3 Further generalization of RSK

Let $|\nu| = k$ be a Young diagram. Let

$$P_\nu(\bar{a}, \bar{b}) = \left\{ f : \nu \rightarrow \mathbb{R}_+ \text{ s.t. } f(i, j) \leq f(i, j+1), f(i+1, j), \sum_j f(i, j) = a_i, \sum_i f(i, j) = b_j \right\}$$

$$Q_\nu(\bar{a}, \bar{b}) = \left\{ g : \nu \rightarrow \mathbb{R}_+ \text{ s.t. } \sum_{i-j=c} g(i, j) = d_c \forall c \right\},$$

where d_c is something.

Theorem 1.3. *There exists Φ^* sending $P_\nu(\bar{a}, \bar{b}) \rightarrow Q_\nu(\bar{a}, \bar{b})$ which is piecewise linear, continuous, volume preserving and a bijection $\Phi^* : P_\nu \cap \mathbb{Z}^k \rightarrow Q_\nu \cap \mathbb{Z}^k$.*

Math 206B Lecture 11 Notes

Daniel Raban

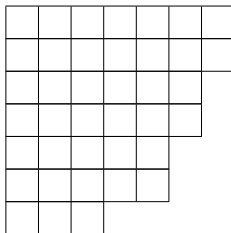
February 1, 2019

1 RSK: The Final Chapter

1.1 Diagonals in Young diagrams and generalizing geometric RSK

Let ν be a partition of k , and let $f : \nu \rightarrow \mathbb{R}_+$ be a function on squares of the Young diagram. Define the diagonal sums $\alpha_c := \sum_{i-j=c} f(i, j)$, where $c \in \mathbb{Z}$. Similarly, define $\beta_c := \sum_{i=1}^{r_c} \sum_{j=1}^{s_c} f(i, j)$, where $c \in \mathbb{Z}$.

Example 1.1. Consider the following partition.



Then

$$\alpha_{-1} = f(1, 2) + f(2, 3) + f(3, 4) + f(4, 5)$$

and $(r_c, s_c) = (5, 4)$ is the position of the lowermost square on this diagonal.

Theorem 1.1. Fix \bar{d} , and let

$$P_\nu(\bar{d}) = \{f : \nu \rightarrow \mathbb{R}_+ \mid f(i, j) \leq f(i+1, j) \leq f(i, j+1), \alpha_c(f) = d_c \forall c\},$$

$$Q_\nu(\bar{d}) = \{g : \nu \rightarrow \mathbb{R}_+ \mid \beta_c(g) = d_c \forall c\}.$$

Then there exists some $\Phi : P_\nu(\bar{d}) \rightarrow Q_\nu(\bar{d})$ such that Φ is

1. piecewise linear,
2. volume-preserving,

3. continuous,

4. $\Phi : P_\nu \cap \mathbb{Z}^L \rightarrow Q_\nu \cap \mathbb{Z}^K$.

Moreover, Φ commutes with transposition.

Corollary 1.1. *The number of integer points in $P_\nu(\bar{d})$ is the same as the number of integer points in $Q_\nu(\bar{d})$.*

Corollary 1.2 (reduction to RSK).

$$\#M(\bar{a}, \bar{b}) = \sum_{\lambda} \# \text{SSYT}(\lambda, \bar{a}) \times \# \text{SSYT}(\lambda, \bar{b})$$

Example 1.2. Let $\nu(\ell^\ell)$ be an $\ell \times \ell$ square. If we split it up along the diagonal, we get two tableaux, a $\text{SSYT}(\lambda, (d_0 - d_{-1}, d_{-1} - d_{-2}, \dots))$ and a $\text{SSYT}(\lambda, (d_0 - d_1, d_1 - d_2, \dots))$. So in the case of a square, we get RSK.

1.2 Description and proof of generalized geometric RSK

Let's prove the theorem.

Proof. Proceed by induction. If $\lambda = \emptyset$, we are done, and if λ is a square, we are also done because P and Q are the same. Let $r - s = c$, so $(r_c, s_c) = (r, s)$. And let $\bar{\nu} = \nu - (r, s)$; we are removing a box from the diagram at position (r, s) on the boundary of the diagram. We have $\Phi_{\bar{\nu}} : P_{\bar{\nu}} \rightarrow Q_{\bar{\nu}}$, and we want to get Φ_ν .

Draw a diagonal from the square (r, s) up and left. We want to alter boxes on the diagonal. Take ξ sending $f(i, j) \mapsto \max\{f(i-1, j), f(i, j-1)\} + \min\{f(i, j+1), f(i+1, j)\} - f(i, j)$. Call this $\bar{f}(i, j)$. Then we get $f \mapsto \bar{f} \in P_{\bar{\nu}} \mapsto \bar{g} \in Q_{\bar{\nu}}$. How do we get $g \in Q_\nu$ from \bar{g} ? Just add the last square by setting $g(r, s) := \bar{f}(r, s) - \max\{\bar{f}(r-1, s), \bar{f}(r, s-1)\}$.

Why is Φ well-defined? Note that no two adjacent diagonals can contain corners. Now think about the order of the squares we chop off and replace. If we write this order in reverse, we get a Young tableau. We claim that if Γ is a graph on $\text{SYT}(\nu)$ with $(i, i+1)$ swaps allowed in distinct diagonals, then Γ_ν is connected. If we keep switching to put every number in our tableau in lexicographic order, we will eventually get the full lexicographic ordering. So the graph Γ_ν is connected, which makes this process well-defined. \square

Example 1.3. Take the element of P_ν

1	1	4
2	3	4
4	4	5

Chop off the 5 in the bottom right hand corner, and alter the diagonal of that 5. After replacing the space of the 5, we get

0	1	4
2	3	4
4	4	1

Now chop off the 4 on the right in the bottom row and alter its diagonal. After replacing that 4, we get

0	1	4
1	3	4
4	0	1

Continue like this, replacing one square at a time in the corner (of the diagram, only counting squares we haven't chopped off and replaced) and altering its diagonal until we've altered everything.

0	2	4
1	3	0
4	0	1

1	2	4
1	1	0
4	0	1

1	2	2
1	1	0
4	0	1

Continuing like this, we eventually get

1	1	2
0	1	0
3	0	1

Math 206B Lecture 12 Notes

Daniel Raban

February 4, 2019

1 RSK: The (final) Final Chapter

1.1 Properties of generalized RSK

Theorem 1.1. $\Phi : P_\nu(\bar{d}) \rightarrow Q_\nu(\bar{d})$ is

1. *piecewise linear,*
2. *volume-preserving,*
3. *continuous,*
4. $\Phi : P_\nu \cap \mathbb{Z}^K \rightarrow Q_\nu \cap \mathbb{Z}^K.$

We had $\Phi = \xi_{(r_n, s_n)} \circ \xi_{(r_{n-1}, s_{n-1})} \circ \cdots$, where $\xi_{(r_i, s_i)}$ is the PL map corresponding to removal of the $(n+1-i)$ -th square in $A \in \text{SYT}(\nu)$. Last time, we showed that Φ is well-defined.

Proof. Invertibility of Φ follows from construction, since we can reverse the process.

To prove that Φ is piecewise linear, let ξ send $f(i, j) \mapsto \max\{f(i-1, j), f(i, j-1)\} + \min\{f(i+1, j), f(i, j+1)\} - f(i, j)$. Then ξ sends $f \mapsto \bar{f}$, so this is also volume preserving, as it has determinant ± 1 .

We now have to show that $\Phi : P_\nu(\bar{d}) \rightarrow Q_\nu(\bar{d})$; that is, if $M \in P_\nu(\bar{d})$ and $\alpha_c(M) = d_c$, then $\Phi(M) \in Q_\nu(\bar{d})$, and $\beta_c(\Phi(M)) = d_c$ for all c . We also need that if $M = [m_{i,j}]$ and $\Phi(M) = [m'_{i,j}]$, then if $m_{i,j} \geq 0$ and $m_{i,j} \leq m_{i,j+1}, m_{i+1,j}$, then $m'_{i,j} \geq 0$. Let's show this 2nd part. We only need to understand this for when we change a corner:

$$\bar{f}(r, s) = f(r, s) - \max\{f(r-1, s), f(r, s-1)\} \geq 0.$$

To show that $\Phi : P_\nu(\bar{d}) \rightarrow Q_\nu(\bar{d})$, proceed by induction on each step. Suppose $\xi_{r,s}$ sends $M \mapsto \bar{M}$, where \bar{M} has share $\nu - (r, s)$. We only alter boxes on the diagonal, so $\alpha_c(M) = \alpha_c(\bar{M})$ for all $c \neq r-s$. Now let $u = r-s$. Then

$$\alpha_u(\bar{M}) = \alpha_{u+1}(M) + \alpha_{u-1}(M) - \alpha_u(M) + m'_{r,s}.$$

We also get, using inclusion-exclusion, that

$$\begin{aligned}\beta_c(M') &= \beta_{c+1}(\Phi(M)) + \beta_{c-1}(\Phi(M)) - \beta_c(\Phi(M) - (r, s)) + m'_{r,s} \\ &= \alpha_{c+1}(M) + \alpha_{c-1}(M) - \alpha_{c-1}(M) + m'_{r,s} \\ &= \alpha_u(\Phi(M)).\end{aligned}$$

Comparing these two equalities proves the property by induction. \square

1.2 Maximal sum over a path

Proposition 1.1. *Let (r, s) be a corner of ν . Then $m_{r,s}$ is equal to the maximal sum over a path from $(1, 1)$ to (r, s) in M' .*

Proof. Look at the formula

$$m'_{r,s} = \bar{f}(r, s) = f(r, s) - \max\{f(r-1, s), f(r, s-1)\}.$$

Then we can prove this by induction. \square

When ν is a square, this is the corresponding property for RSK.

What is the moral here? We generalized RSK so far that we can prove all these properties from just two equations involving α and β . Later, we will approach RSK from a different angle, involving f^λ .

Math 206B Lecture 14 Notes

Daniel Raban

February 8, 2019

1 Applications of The Hilman-Grassl Bijection

1.1 MacMahon's theorem

We showed the following last time:

Theorem 1.1 (Stanley).

$$\sum_{A \in \text{RPP}(\lambda)} t^{|A|} = \prod_{(i,j) \in \lambda} \frac{1}{1 - t^{h_{i,j}}},$$

where $h_{i,j}$ is the hook length of the square i, j in the Young diagram of λ .

Definition 1.1. A **plane partition** of shape λ is a tableau which is weakly decreasing going down or to the right.

Example 1.1. Here is a plane partition of shape $(4, 4, 3, 2)$.

7	7	6	4
4	3	1	1
3	3	1	
2			

Theorem 1.2 (MacMahon, c. 1900).

$$\sum_{A \in \text{PP}} t^{|A|} = \prod_{k=1}^{\infty} \frac{1}{(1 - t^k)^k}$$

Proof. Let $\lambda = m^m$; this is a square diagram. Then Stanley's theorem says

$$\sum_{A \in \text{RPP}(m^m)} t^{|A|} = \prod_{k=1}^m \frac{1}{(1 - t^k)^k} \cdot \prod_{k=m+1}^{2m-1} \frac{1}{(1 - t^k)^{2m-k}}.$$

Now

$$\sum_{A \in PP} t^{|A|} = \lim_{m \rightarrow \infty} \sum_{A \in PP \cap (m^m)} t^{|A|}$$

because this limit stabilizes for each coefficient of the power series.

$$\begin{aligned} &= \lim_{m \rightarrow \infty} \sum_{A \in RPP(m^m)} t^{|A|} \\ &= \lim_{m \rightarrow \infty} \prod_{k=1}^m \frac{1}{(1-t^k)^k} \cdot \prod_{k=m+1}^{2m-1} \frac{1}{(1-t^k)^{2m-k}} \\ &= \prod_{k=1}^m \frac{1}{(1-t^k)^k}. \end{aligned}$$

□

MacMahon's theorem is analogous to the following result.

Theorem 1.3 (Euler, 1738).

$$\sum_{\lambda \in P} t^{|\lambda|} = \prod_{k=1}^{\infty} \frac{1}{1-t^k}.$$

1.2 The hook length formula

We can also prove the hook length formula using Stanley's theorem.

Theorem 1.4 (Frame-Robinson-Thrall, 1954).

$$f^{\lambda} = \# \text{SYT}(\lambda) = n! \prod_{(i,j) \in \lambda} \frac{1}{h_{i,j}}.$$

Example 1.2. Suppose $\lambda = (m, m)$. Here is a standard Young tableau of this shape:

1	2	4	5	9
3	6	7	8	10

Then the hook lengths of each square look like

$m+1$	m	\cdots	3	2
m	$m-1$	\cdots	2	1

In general,

$$f^{(m,m)} = \frac{(2m)!}{m!(m-1)!} = \frac{1}{m+1} \binom{2m}{m} = \text{Cat}(m),$$

the m -th Catalan number.

Proof. Write

$$\text{RPP}(\lambda) = \bigcup_{A \in \text{SYT}(\lambda)} C_A,$$

where C_A is where we pick the numbers in order of sums of indices in the diagram. That is $0 \leq x_{1,1} \leq x_{1,2} \leq x_{2,1} \leq x_{1,3} \leq x_{2,2} \leq x_{1,4} \leq x_{2,3} \leq x_{3,1} \leq \dots$; this is a cone in \mathbb{R}^n . Now look at the number of $A \in \text{RPP}(\lambda)$ such that $|A| \leq N$. Asymptotically, this is about

$$\begin{aligned} \sum_{T \in \text{SYT}(\lambda)} \#\{A \in C_T : |A| \leq N\} &= \#\text{SYT}(\lambda) \cdot \#\{0 \leq z_1 \leq \dots \leq z_n : z_1 + \dots + z_n \leq N\} \\ &\sim \#\text{SYT}(\lambda) \cdot N^n \text{vol}(\Delta), \end{aligned}$$

where $\Delta = \{0 \leq z_1 \leq z_2 \leq \dots \leq z_n : z_1 + \dots + z_n \leq 1\}$. The vertices of Δ are when we have equalities. So we have

$$\begin{aligned} v_0 &= (0, 0, \dots, 0) \\ v_1 &= (0, \dots, 0, 1) \\ v_2 &= (0, \dots, 0, 1/2, 1/2) \\ v_3 &= (0, \dots, 0, 1/3, 1/3, 1/3) \\ &\vdots \\ v_n &= (1/n, \dots, 1/n). \end{aligned}$$

This is a triangular matrix, so the volume ($1/n!$ times the determinant), is $1/n!$ times the product of the diagonal entries. That is,

$$\text{Vol}(\Delta) = \frac{1}{n!} \cdot \frac{1}{n!}.$$

So we get

$$|\{A \in \text{RPP}(\lambda) : |A| \leq N\}| \sim \frac{N^n}{(n!)^2} \#\text{SYT}(\lambda).$$

The right hand side of Stanley's theorem is

$$\sum t^{\sum b_{i,j} h_{i,j}},$$

where the sum is over matrices $B = (b_{i,j})$ such that $b_{i,j} \geq 0, \sum b_{i,j} \leq N$. This is asymptotically $N^n \text{vol}(\Delta')$, where $\Delta' = \{0 \leq y_{i,j}, \sum y_{i,j} h_{i,j} \leq 1\}$. The vertices of Δ' are

$$\begin{aligned} &(0, 0, \dots, 0) \\ &(1/h_{1,1}, 0, \dots, 0) \\ &(0, 1/h_{1,2}, 0, \dots, 0) \end{aligned}$$

\vdots

So we get that

$$\text{Vol}(\Delta') = \frac{1}{n!} \cdot \prod_{(i,j) \in \Delta} \frac{1}{h_{i,j}}.$$

By Stanley's theorem, we have

$$|\{A \in \text{RPP}(\lambda) : |A| \leq N\}| = |\{B : b_{i,j} \geq 0, \sum b_{i,j} \leq N\}|$$

So the asymptotics have to be the same. Then we get that

$$\# \text{SYT}(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}. \quad \square$$

We will prove the hook length formula in different ways, as well.

Math 206B Lecture 15 Notes

Daniel Raban

February 11, 2019

1 The NPS Algorithm

1.1 Motivation

Here is the hook length formula.

Theorem 1.1 (FRT, 1958).

$$f^\lambda = \#\text{SYT}(\lambda) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}}.$$

The following is a general result from representation theory.

Theorem 1.2. *Let χ be a character of a finite group G , and let $d = \chi(1)$ be the dimension of the representation. Then $d \mid |G|$.*

In this case, we get the following.

Corollary 1.1. $\#\text{SYT}(\lambda) \mid n!$.

Why should this be true?

Example 1.1. Let $\lambda = (n - k, 1^k)$. Then

$$f^\lambda = \binom{n-1}{k} = \frac{(n-1)!}{k!(n-k)!} \mid (n-1)!,$$

so it divides $n!$.

Example 1.2. Let $\lambda = (m, m)$, where $n = 2m$. Then

$$f^\lambda = \frac{1}{m+1} \binom{2m}{m} = \frac{(2m)!}{m!(m+1)!} = \frac{n!}{m!(m+1)!}.$$

Why should this be an integer?

Today, we will construct $\Phi_\lambda : S_n \rightarrow \text{SYT}(\lambda)$, such that $\Phi_\lambda^{-1}(A)$ is a constant depending on λ . In reality, this constant will be $\prod h_{i,j}$.

Example 1.3. Let $\lambda = (n)$. Then $f^\lambda = 1$. In this case, Φ_λ will be some sorting algorithm.

Example 1.4. Let $\lambda = (1^n)$. Then $f^\lambda = 1$. In this case, Φ_λ will sort vertically.

1.2 Construction of Φ_λ

This will sort of be like 2-dimensional bubble sort.

Take $\lambda = (5, 5, 3, 2)$, and choose a permutation:

9	8	11	12	4
15	3	14	2	5
10	7	1		
13	6			

Look at the last column. The column is sorted, so we are ok. Now include the next column. 12 is bigger than 2, so we must switch them. Same with 12 and 5.

9	8	11	2	4
15	3	14	5	12
10	7	1		
13	6			

Now look at the 1 in the next column. This is sorted. Now look at the 14. This is bigger than the 1, so we need to switch it.

9	8	11	2	4
15	3	1	5	12
10	7	14		
13	6			

Going up the columns, we need to switch the 11 with something. It must be switched with the 1. Then we need to switch the 11 with the 5.

9	8	1	2	4
15	3	5	11	12
10	7	14		
13	6			

Move on to the next column. The 6 is fine, but when we move up to the 7, we see that we have to switch it with the 6.

9	8	1	2	4
15	3	5	11	12
10	6	14		
13	7			

The 3 is fine where it is. But the 8 above needs to be switched. We switch it with the 1, then the 2, and then the 4 until this part is sorted.

9	1	2	4	8
15	3	5	11	12
10	6	14		
13	7			

The 13 is okay, but the 10 needs to be switched. Switch it with the 6.

9	1	2	4	8
15	3	5	11	12
6	10	14		
13	7			

The 15 has a long way to go. See if you can figure out where it needs to go:

9	1	2	4	8
3	5	11	12	15
6	10	14		
13	7			

Finally, we move the 9 where it needs to go:

1	2	4	8	9
3	5	11	12	15
6	10	14		
13	7			

We now have a standard Young tableau. We had no choice at each step, so we can see that this algorithm is well-defined. Notice that this is similar to jeu-de-taquin.

1.3 Construction of Ψ_λ

Now we will construct $\Psi_\lambda : S_n \rightarrow \prod_{(i,j) \in \lambda} [-\lambda'_j + 1, \dots, \lambda_i - i]$. This range has size $h_{i,j}$.

Lemma 1.1. $(\Phi_\lambda, \Psi_\lambda)$ gives a bijection between S_n and the Cartesian product of $\text{SYT}(\lambda)$ with the above product.

Example 1.5. When we are just doing bubblesort, Ψ_λ will give us (a_1, \dots, a_2, a_1) , where a_i is the number of steps the i -th number made.

Take our example from earlier:

9	8	11	12	4
15	3	14	2	5
10	7	1		
13	6			

Let's record the sorting procedure and keep track of what numbers move.

9	8	11	12	4	0	0	0	0	0
15	3	14	2	5	0	0	0	0	0
10	7	1			0	0	0		
13	6				0	0			

12 is moved down and then to the right. Our rule is that when we move to the right, we add 1. When we move a number down, we switch the two numbers and subtract 1 from the top.

9	8	11	2	4	0	0	0	-1	0
15	3	14	12	5	0	0	0	0	0
10	7	1			0	0	0		
13	6				0	0			

9	8	11	2	4	0	0	0	-1	0
15	3	14	5	12	0	0	0	1	0
10	7	1			0	0	0		
13	6				0	0			

When we move the 14 down, we get:

9	8	11	2	4	0	0	0	-1	0
15	3	1	5	12	0	0	-1	1	0
10	7	14			0	0	0		
13	6				0	0			

When 11 goes down, we switch the two blocks and then subtract 1 from the new top:

9	8	1	2	4
15	3	11	5	12
10	7	14		
13	6			

0	0	-2	-1	0
0	0	0	1	0
0	0	0		
0	0			

Now the 11 has to move to the right:

9	8	1	2	4
15	3	5	11	12
10	7	14		
13	6			

0	0	-2	-1	0
0	0	1	1	0
0	0	0		
0	0			

The 7 in the next column has to be moved down:

9	8	1	2	4
15	3	5	11	12
10	6	14		
13	7			

0	0	-2	-1	0
0	0	1	1	0
0	-1	0		
0	0			

Now we have to move the 8. We only change the numbers in the “column and row of action.”

9	1	2	4	8
15	3	5	11	12
10	6	14		
13	7			

0	3	-2	-1	0
0	0	1	1	0
0	-1	0		
0	0			

Now move the 13:

9	1	2	4	8
15	3	5	11	12
10	6	14		
13	7			

0	3	-2	-1	0
0	0	1	1	0
0	-1	0		
1	0			

Continuing like this, we get

Math 206B Lecture 16 Notes

Daniel Raban

February 13, 2019

1 Hook Walks and Representation Theory of Stanley's Formula

1.1 Hook walks

We want to prove the hook length formula in a more elegant way. This is a construction due to Greene, Nijenhuis, and Wilf in 1979.¹

We will probabilistically construct a standard Young tableau. First, take the empty diagram. We need to put n into one of the corner spaces. We get

$$\mathbb{P}(n \text{ is at } (r, s)) = \frac{f^{\lambda^*}}{f^\lambda} \stackrel{\text{WTS}}{=} \frac{(n-1)!}{n!} \prod \frac{h_{i,j}}{h_{i,j}^*},$$

where λ^* is $\lambda - (r, s)$. Using this, if we had a guess for the hook length formula, we could try to prove the hook length formula by induction. However, this is actually not that easy. So GNW constructed a random process to do this.

The idea is a **hook walk**.

1. Start at a random $(i, j) \in \lambda$.
2. Mover to a random square in $\text{Hook}_\lambda(i, j)$.
3. Repeat step 2 until you get to the corner.

Lemma 1.1. $\mathbb{P}(\text{corner is at } (r, s)) = f^{\lambda^*} / f^\lambda$.

Given this lemma, using the fact that $\sum \mathbb{P}(\text{corner is at } (r, s)) = 1$, we get $f^\lambda = \sum_{\lambda^*} f^{\lambda^*}$. This lets us use the guess for the hook length formula to use our induction. Here is a stronger result, which is easier to prove.

¹This is actually before the NPS algorithm, which is from 1992.

Lemma 1.2. *Let α_i be the column number of the i -th square in this algorithm, and let β_i be the row number of the i -th square. Then*

$$\mathbb{P}((i, j) \rightarrow (r, s) \text{ via } \bar{\alpha}, \bar{\beta}) = \frac{1}{n} \prod \left(1 + \frac{1}{h_{i, \alpha} 1}\right) \prod \left(1 + \frac{1}{h_{\beta, j} - 1}\right).$$

For the precise statement of the lemma, look at Exercise 3.17 in the textbook (Sagan's *The Symmetric Group*).

1.2 Stanley's formula, explained

Let $W_n = \mathbb{C}[x_1, \dots, x_n]$, thought of as an infinite dimensional representation of S_n . Let $H_n \subseteq W_n$ be the set of **harmonic polynomials**, nonconstant polynomials $f \in W_n$ such that $h \cdot f = 0$ for all $h \in \mathbb{C}[\partial/\partial x_1, \dots, \partial/\partial x_n]^{S_n}$.

Example 1.1. Let $n = 2$. Then what survives when we apply $\partial/\partial x_1, \partial/\partial x_2$ or $\partial/\partial x_1 \partial x_2$? We get that $H_2 = \mathbb{C}\langle 1, x_1 - x_2 \rangle$.

Theorem 1.1 (Chevalley). *Let W_n, H_n be as above.*

1. $W_n = H_n \otimes I_n$, where $I_n = W^{S_n}$ is the symmetric polynomials.
2. H_n is the regular S_n -representation.

Write $W_n = \bigoplus_{k=0}^{\infty} W_n^k$, $H_n = \bigoplus_{k=0}^{\binom{n}{2}} H_n^k$, and $I_n = \bigoplus_{k=0}^{\infty} I_n^k$. This is a grading by the degree of the polynomials. Write

$$P_n(t) = \sum_{k=0}^{\infty} (\dim(W_n^k) t^k) = 1/(1-t)^n$$

$$I_n(t) = \sum_{k=0}^{\infty} (\dim(I_n^k) t^k) = \frac{(1-t)(1-t^2) \cdots (1-t^n)}{1-t^{n+1}}$$

$$H_n(t) = \frac{P_n(t)}{I_n(t)} = \prod_{i=1}^n \frac{1-t^i}{1-t} = \prod_{i=1}^n (1+t+\cdots+t^{i-1}).$$

Now

$$P_{\lambda}(t) = \sum_{k=0}^{\infty} \dim(\text{Hom}(W_n^k, S^{\lambda})) t^k = \prod_{(i,j) \in \text{in } \lambda} \frac{t^{i-1}}{1-t^{h_{i,j}}}.$$

Then

$$H_{\lambda}(t) = \sum_{k=0}^{\infty} \dim(\text{Hom}(H_n^k, S^{\lambda})) t^k = H_n(t) \prod_{(i,j) \in \lambda} \frac{1}{(h_{i,j})_t},$$

the t -analogue of $h_{i,j}$. Then the hook length formula is

$$H_\lambda(1) = f^\lambda = \frac{n!}{\prod h_{i,j}}.$$

This is an algebraic interpretation of how Stanley's formula implies the hook length formula.

Math 206B Lecture 17 Notes

Daniel Raban

February 15, 2019

1 Odds, Ends, and Things to Come

1.1 Recap episode

We have learned a few things so far:

1. Representation theory of S_n
2. Combinatorics of Young tableau and combinatorial algorithms
 - (a) RSK algorithm
 - (b) Hook length formula
 - (c) Stanley's formula and the Hilman-Grassl algorithm.

We will learn about two more things:

1. $GL_N(\mathbb{C})$ representation theory
2. Symmetric functions.

These are related to further topics we won't cover, namely Schubert calculus and enumerative algebraic geometry.

1.2 Combinatorial questions about partitions

Theorem 1.1 (MacMahon, c.1900). *Let $PP(\lambda, c)$ be the set of plane partitions of shape λ numbers $\leq c$. Then $PP(a^b, c)$ (rectangles with numbers at most c) satisfies*

$$|PP(\lambda, c)| = \prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{i+j+k-1}{i+j+k-2}.$$

Here is a q -version.

Theorem 1.2.

$$\sum_{A \in \text{PP}(a^b, c)} q^{|A|} = \prod_{i=1}^a \prod_{j=1}^b \prod_{k=1}^c \frac{(i+j+k-1)_q}{(i+j+k-2)_q},$$

where $(m)_q = (1 - q^m)/(1 - q)$.

So when $q \rightarrow 1$, we get the original theorem.

The number of Lozenge tilings of an $a \times b \times c$ hexagon is the number of $\text{PP}(a^b, c)$. View the tilings as stacked 3-dimensional cubes, a 3-dimensional partition of an $a \times b \times c$ box. This is a bit strange. For any graph G , the number of perfect matching in G is the determinant of some matrix. We can view a tiling as a matching in terms of the dual graph. How can this determinant have a product formula?

1.3 Related ideas in representation theory

Theorem 1.3 (Frobenius, c.1902). *Let ξ^α be the character corresponding to M^α , let $\lambda = (\lambda_1, \dots, \lambda_\ell)$, let $\rho = (\ell, \dots, 1)$ be a staircase partition, and for a permutation ω , let $\omega\alpha = (\alpha_{\omega(1)}, \alpha_{\omega(2)}, \dots)$. Then*

$$\chi^\lambda = \sum_{\omega \in S_\ell} \text{sign}(\omega) \xi^{\lambda - \omega\rho + \rho}.$$

Here, the convention is $\xi^\alpha = 0$ for all $\alpha \notin \mathbb{N}^\ell$ (if we get negative entries, discard the term).

This formula looks a little like a determinant. There is a relationship between these formulas.

Theorem 1.4 (Kostka). *Let $K_{\lambda, \mu} = \# \text{SSYT}(\lambda, \mu)$. Then*

$$\xi^\mu = \sum_{|\lambda|=n} K_{\lambda, \mu} \chi^\lambda.$$

This explains the triangular nature of everything because you cannot have a semistandard Young tableau with shape λ and weight μ if $\lambda \leq \mu$. Frobenius's theorem gives a form for the inverse of the matrix

$$(K_{\lambda, \mu})_{|\lambda|=|\mu|=n}^{-1}.$$

These values are called the **Kostka numbers**. Here is what we will learn:

Definition 1.1. The **Schur functions** are $s_\lambda = \sum_{A \in \text{SSYT}(\lambda)} x_a^{m_1(A)} \dots x_N^{m_N(A)}$, where $m_i(A)$ is the number of i in A .

Theorem 1.5. *The s_λ are symmetric functions ($\mathbb{C}[x_1, \dots, x_N]^{S_N}$).*

Definition 1.2. The **elementary symmetric polynomials** are $e_\alpha = e_{\alpha_1} e_{\alpha_2} \dots$.

Theorem 1.6 (Jacobi-Trudy). *Using the convention that $e_r = 0$ for all $r < 0$,*

$$s_\lambda = \det[\tilde{e}_{\lambda_i + j - i}]_{i,j=1,\dots,\ell}.$$

It turns out that the Jacobi-Trudy theorem is saying the same thing as Frobenius's theorem. Also, the definition of Schur functions is equivalent to Kostka's theorem.¹ The idea is then that

$$|\text{PP}(a^b, c)| = s_{(a,b)}(\underbrace{1, 1, \dots, 1}_c).$$

Since the Schur functions have a determinant formula, we can see why there is a determinant formula for Lozenge tilings. To get the q -analogue, we can look at $s_{(a,b)}(1, q, q^2, \dots, q^{c-1})$.

This is the general picture we will be going through in the next few weeks.

¹This is in the same sense that whoever came up with the definition of variance understood a lot about scalar products. The definition is designed to be compatible with the nice structure.

Math 206B Lecture 18 Notes

Daniel Raban

February 20, 2019

1 Relationships of Schur Functions to Characters of $\mathrm{GL}_N(\mathbb{C})$ and to Young Tableau

1.1 Schur functions and characters of $\mathrm{GL}_N(\mathbb{C})$

Definition 1.1. The ring of **symmetric functions of degree n** is $\mathbb{C}[x_1, \dots, x_n]^{S_n}$. The ring of **symmetric functions** is $\Lambda = \varprojlim_n \Lambda_n$.

Example 1.1. Λ_3 is spanned by $x_1 + x_2 + x_3$, $x_1x_2 + x_1x_3 + x_2x_3$ and $x_1x_2x_3$.

A representation is a homomorphism $\rho : V \rightarrow \mathrm{Aut}(V)$, where $V = \mathbb{C}^d$. Essentially, what we do with representations of finite groups can be done with compact groups, as well.

Theorem 1.1. *All representations of $\mathrm{GL}_N(\mathbb{C})$ are rational. That is, if $M = (x_{i,j})$, then $\rho(M) = (f_{pq}(x_i))$, where the $f_{p,q}$ are rational polynomials. Moreover, $\rho(M) = (\det)^k$ times a polynomial representation.*

Example 1.2. Let ρ be the determinant map sending $M \mapsto \det(M)$. This is a 1-dimensional representation. Then $\rho_k(M) = (\det(M))^k$ is also a representation for $k \in \mathbb{Z}$.

Let $\lambda = \lambda_1 \geq \dots \geq \lambda_N$ with $\lambda \in \mathbb{N}$. If ρ is an irreducible representation of $\mathrm{GL}_N(\mathbb{C})$, then consider diagonal matrices M and “characters of ρ ” given by $\mathrm{tr}(\rho[M])$. Then these are in Λ_N .

Theorem 1.2 (Weyl¹). *If π is an irreducible representation of $\mathrm{GL}_N(\mathbb{C})$ corresponding to $\lambda = (\lambda_1 \geq \dots \geq \lambda_N)$, then $\mathrm{tr}(\pi) = s_\lambda$, where $s_\lambda = a_{\lambda+\rho}/a_\rho$ is a Schur function.*

Here, if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $a_\alpha = \det(x_i^{\alpha_j})_{i,j=1,\dots,N}$ is like a Vandermonde determinant, and $\rho = (N-1, N-2, \dots, 0)$ (so a_ρ is a Vandermonde determinant).

Example 1.3. Let $N = 2$ and $\lambda = (4, 3)$. Then

$$s_\lambda = \frac{\det \begin{bmatrix} x^4 & y^4 \\ x^3 & y^3 \end{bmatrix}}{x - y} = \frac{x^4y^3 - y^4x^3}{x - y} = x^3y^3.$$

¹Igor does not remember whose theorem this is, so Weyl is a guess.

Remark 1.1. The situation in the theorem is actually something that happens for all compact groups.

1.2 Formula for Schur functions in terms of Young tableau

Theorem 1.3. Let $\lambda = (\lambda_1 \geq \dots \geq \lambda_N)$. Then

$$s_\lambda = \sum_{A \in \text{SSYT}(\lambda, \leq N)} x_1^{m_1(A)} \dots x_N^{m_N(A)},$$

where $m_i(A)$ is the number of i s in A .

Lemma 1.1. This sum is a symmetric polynomial of degree N .

Proof. We show that the sum is invariant under the transposition $(i \ i+1) \in S_N$ for all $i = 1, \dots, N-1$. Given a semistandard Young tableau, look at each row. If we look at rows with i and $i+1$, look at parts where we do not have $i+1$ squares directly below i squares. Then we can switch the number of i s and $(i+1)$ s in this part of each row and still get a semistandard Young tableau. \square

Proof. We can prove the theorem by showing that a_ρ times the sum is $a_{\lambda+\rho}$.

$$\prod_{1 \leq i < j \leq N} (x_i - x_j) = \sum_{\sigma \in S_N} \text{sign}(\sigma) s_{\sigma(1)}^0 x - \sigma(2)^1 \dots x_{\sigma(N)}^{N-1}$$

Here is Gessel's proof of this fact. It is easier to consider $\prod_{i < j} (1 - x_j/x_i)$. \square

Math 206B Lecture 19 Notes

Daniel Raban

February 22, 2019

1 Bases, Involution, and Scalar Product of Symmetric Functions

1.1 Five bases of symmetric functions

Let $\Lambda = \varprojlim_n \lambda_n$ be the ring of symmetric functions. Then $\Lambda \subseteq \mathbb{C}[x_1, x_2, \dots]$; that is $f \in \Lambda = \sum c_\alpha x^\alpha$, where $\alpha = (\alpha_1, \alpha_2, \dots)$, $\sum \alpha_i < \infty$, and $\alpha_i \in \mathbb{N}$.

Example 1.1. $e_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 + x_1x_5 + \dots$.

Definition 1.1. The **monomial symmetric functions** are

$$m_\lambda = \left[\sum_{\substack{\sigma \in S_\ell \\ i_1 < \dots < i_\ell}} x_{i_1}^{\lambda_{\sigma(1)}} x_{i_2}^{\lambda_{\sigma(2)}} \dots x_{i_\ell}^{\lambda_{\sigma(\ell)}} \right] / \prod_{i=1}^{\ell} m_i(\lambda)!,$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 1.1. *The monomial symmetric functions form a basis for Λ .*

Definition 1.2. The **elementary symmetric functions** are

$$e_k = m_{(1^k)} = \sum_{i_1 < \dots < i_k} x_{i_1} \dots x_{i_k},$$
$$e_\lambda = e_{\lambda_1} \dots e_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 1.2. *The elementary symmetric functions form a basis for Λ .*

Proof. $f \in \Lambda$ is $c_\lambda x^\lambda + \dots$, when written in lexicographic order. □

Theorem 1.1. *The elementary symmetric functions are free generators of Λ as a ring; i.e. they do not satisfy any algebraic equations.*

Definition 1.3. The **power symmetric functions** are

$$p_k = m_{(k)} = x_1^k + x_2^k + \cdots,$$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 1.3. *The power symmetric functions form a basis for Λ .*

Definition 1.4. The **complete symmetric functions** are

$$h_k = \sum_{|\lambda|=k} m_\lambda = \sum_{i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

$$h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell},$$

where $\lambda = (\lambda_1, \dots, \lambda_\ell)$.

Proposition 1.4. *The complete symmetric functions form a basis for Λ .*

Definition 1.5. The **Schur functions** are

$$s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} = \sum_{A \in \text{SSYT}(\lambda)} x^A,$$

$$x^A = x_1^{m_1(A)} x_2^{m_2(A)} \cdots.$$

Theorem 1.2. *The Schur functions form a basis for Λ .*

1.2 Involution and scalar product on symmetric functions

Here is a dictionary relating symmetric functions and representation theory of S_n

Symmetric functions	Representations of S_n
s_λ	S^λ
h_λ	$M^\lambda = \text{ind}_{S_{\lambda_1} \times \cdots \times S_{\lambda_\ell}}^{S_n} 1$
e_λ	$M^\lambda \otimes \text{sgn}$

This correspondence tells that we should have an involution $\omega : \Lambda \rightarrow \Lambda$ sending $e_\lambda \mapsto h_\lambda$ corresponding to $\otimes \text{sgn}$.

Theorem 1.3. *The involution $\omega : \Lambda \rightarrow \Lambda$ sends $s_\lambda \mapsto s_{\lambda'}$.*

Theorem 1.4. *The involution $\omega : \Lambda \rightarrow \Lambda$ sends $p_\lambda \mapsto \varepsilon_\lambda p_\lambda$.*

There is a scalar product on Λ that relates to the scalar product on characters of representations of S_n .

Definition 1.6. Define a **scalar product** on Λ by its value on the basis of Schur functions:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda, \mu}.$$

If $f = \sum c_\lambda s_\lambda$ and $g = \sum r_\lambda s_\lambda$, then

$$\langle f, g \rangle = \sum_\lambda c_\lambda r_\lambda.$$

Proposition 1.5. $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda, \mu}$ for all λ, μ .

Proof. Write $M^\mu = \bigoplus K_{\lambda, \mu} S^\lambda$. Then $h_\mu = \sum K_{\lambda, \mu} s_{\lambda, \mu}$. We also have $s_\lambda = \sum K_{\lambda, \mu} m_\mu$. \square

Proposition 1.6. $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda, \mu}$, where

$$z_\lambda = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \dots}$$

is the size of the conjugacy class corresponding to λ in S_n .

Theorem 1.5. $s_\lambda = \sum_\mu \chi_\lambda[\mu] p_\mu$.¹

So we can talk about characters of the symmetric group by only talking about symmetric functions.²

¹Maybe there is a factor of z_λ in here. Professor Pak doesn't remember.

²Newton studied p_μ . That's how old the idea of symmetric functions is.

Math 206B Lecture 20 Notes

Daniel Raban

February 25, 2019

1 Generating Functions of Symmetric Functions and Cauchy-Type Identities

1.1 Relationship between e_k and h_k

We want to prove a correspondence between e_n and h_n . Define $E(t) = \sum_{n=0}^{\infty} e_n t^n$ and $H(t) = \sum_{n=0}^{\infty} h_n t^n$. So we just need to prove a relation between E and H .

Proposition 1.1. $E(t) = \prod_{i=1}^{\infty} (1 + x_i t)$.

Proof. This follows from $e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}$.¹ □

Proposition 1.2. $H(t) = \prod_{i=1}^{\infty} \frac{1}{1 - x_i t}$.

Proof. This follows from $h_n = \sum_{i_1 \leq \dots \leq i_n} x_{i_1} \cdots x_{i_n}$. □

Corollary 1.1. $E(t)H(-t) = 1$.

Definition 1.1. The ω -**involution** $\omega : \Lambda \rightarrow \Lambda$ is $\omega(e_k) = h_k$ for all k .

Proposition 1.3. $\omega^2 = \text{id}_{\Lambda}$.

Proof. This is equivalent to $\omega(h_k) = e_k$ for all k . The relation for E and H shows that $\sum_{i=0}^n (-1)^{n-i} e_i h_{n-i} = 0$ for all n . The map ω is an algebra homomorphism, so applying ω to the sum should still give 0. We can then recursively determine that $\omega(h_k) = e_k$. □

Example 1.1. Say we are looking at n variable symmetric functions. Then

$$e_k(1, \dots, 1) = \binom{n}{k},$$
$$h_k(1, \dots, 1) = \binom{n+k-1}{k}.$$

¹Igor says that this is the kind of thing he thinks is so simple, he doesn't actually know how to really prove it. He's not being facetious.

Some people call this latter quantity $\binom{-n}{k}$. The generating function of $\binom{n}{k}$ is

$$\sum_{k=0}^{\infty} \binom{n}{k} t^k = (1+t)^n,$$

while the generating function of $\binom{-n}{k}$ is

$$\sum_{k=0}^{\infty} \binom{-n}{k} t^k = (1-t)^{-n}.$$

So this is the symmetric function analogue of these classic generating function identities.

1.2 Cauchy-type identities

Consider the multivariate generating function

$$Q(\bar{x}, \bar{y}) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}.$$

Proposition 1.4.

$$Q(\bar{x}, \bar{y}) = \sum_{\alpha, \beta} |\text{Mat}(\alpha, \beta)| m_{\alpha}(x) m_{\beta}(y) = \sum_{\lambda} m_{\lambda}(x) h_{\lambda}(y),$$

where $\text{Mat}(\alpha, \beta)$ are matrices with row-sums α and column-sums β .

Proof. For the first part, the $x^{\alpha} y^{\beta}$ term in the product is a sum of terms of products of $(x_i y_j)^{a_{i,j}}$. If we put the $a_{i,j}$ into a matrix, the matrix has row-sums α and column-sums β .

For the second part, use $h_{\lambda} = \sum_{\alpha} |\text{Mat}(\alpha, \lambda)| m_{\alpha}(x)$. \square

Proposition 1.5.

$$Q(\bar{x}, \bar{y}) = \sum_{\lambda} z_{\lambda} p_{\lambda}(\bar{x}) p_{\lambda}(\bar{y}).$$

Theorem 1.1.

$$Q(\bar{x}, \bar{y}) = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y).$$

Proof. This follows from RSK. The coefficient of $x^{\alpha} y^{\beta}$ in $s_{\lambda}(x) s_{\lambda}(y)$ is $|\text{SSYT}(\lambda, \alpha)| \cdot |\text{SSYT}(\lambda, \beta)|$. \square

When Richard Stanley wrote his textbook, he didn't believe that Cauchy proved this identity. After looking through hundreds of pages of Cauchy's works, he determined that Cauchy did not actually prove this identity. After more research, he found that Cascoux was the first person to cite the identities as theorems of Cauchy. So in fact, none of these Cauchy-type identities are due to Cauchy.

Proposition 1.6. $\{f_\alpha\}$ is an orthonormal basis of Λ if

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} f_{\lambda}(x) f_{\lambda}(y).$$

Corollary 1.2. $\{p_{\lambda}/\sqrt{z_{\lambda}}\}$ is an orthonormal basis of λ .

Math 206B Lecture 21 Notes

Daniel Raban

February 27, 2019

1 The Jacobi-Trudi Identity

1.1 Connection to Frobenius' theorem

Theorem 1.1 (Jacobi-Trudy¹). *Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$. Then*

$$s_\lambda = \det(h_{\lambda_1 - i + j})_{i,j=1}^\ell.$$

This should remind you of the following theorem:

Theorem 1.2 (Frobenius).

$$\chi^\lambda = \sum_{\omega \in S_\ell} \text{sgn}(\omega) \zeta^{\lambda + \omega \rho - \rho},$$

where ζ^μ is the character of M^μ and $\rho(\ell - 1, \ell - 2, \dots, 1, 0)$.

In fact, these are the same.

Example 1.1. Let $\lambda = (4, 2, 1)$, so $\ell = 3$. Then

$$s_{(4,3,2)} = \begin{vmatrix} h_4 & h_5 & h_6 \\ h_1 & h_2 & h_3 \\ 0 & 1 & h_1 \end{vmatrix} = h_{(4,2,1)} - h_{(5,1,1)} - h_{(4,3)} + h_{(6,1)}$$

The Frobenius formula in this case says

$$\begin{aligned} \chi^{(4,2,1)} &= \zeta^{(4,2,1)-(2,1,0)+(2,1,0)} - \zeta^{(4,2,1)-(1,2,0)+(2,1,0)} \\ &\quad - \zeta^{(4,2,1)-(2,0,1)+(2,1,0)} + \zeta^{(4,2,1)-(0,2,1)+(2,1,0)} + 0 + 0 \\ &= \zeta^{(4,2,1)} - \zeta^{(5,1,1)} - \zeta^{(4,3)} + \zeta^{(6,1)}. \end{aligned}$$

¹You can find this in section 7.16 of Richard Stanley's Enumerative Combinatorics volume 2.

1.2 Proof of the identity

The idea of the proof is term cancellation. We expand the determinant and show that a lot of terms cancel. Here is an analogy.

Proposition 1.1. *Let $A = (a_{i,j})$, $B = (b_{i,j})$ be $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.*

You prove this by representing both sides as counting something and showing that the extra terms in $\det(AB)$ compared to $\det(A)\det(B)$ cancel out in pairs.

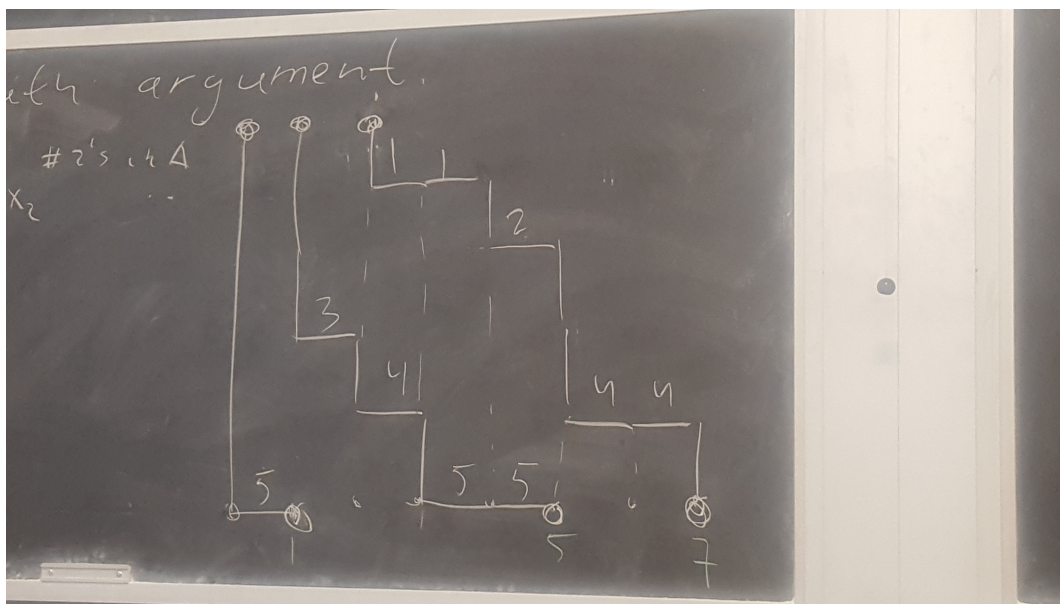
Proof. The idea is a non-crossing path argument, like you can use to prove the statement about determinants.

$$S_\lambda = \sum_{A \in \text{SSYT}(\lambda)} x^A = \sum_A x_1^{\#1s} x_2^{\#2s} \dots$$

Take A and construct a system of paths. For example, if

$$A = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 4 & 4 \\ \hline 3 & 4 & 5 & 5 & \\ \hline 5 & & & & \\ \hline \end{array}$$

construct the system of paths



Fix the starting points and ending points. Now consider all systems of paths from the starting points to the ending points. A path Q will produce the monomial $x^Q =$

$x_1^{\#1s} x_2^{\#2s} \cdots$, and if we switch two paths that intersect, we get a $\text{sgn}(\sigma)$ coefficient in front. Then each system with a pair of paths that intersect somewhere will not be counted because we can just switch the paths after the intersection point; in this case, the contribution of the paths to the sum will cancel. Since the paths define polynomials h_λ with this definition, we are done. \square

Math 206B Lecture 22 Notes

Daniel Raban

March 1, 2019

1 Littlewood-Richardson Coefficients

1.1 Multiplying symmetric functions

Recall

$$s_\lambda = \sum_{A \in \text{SSYT}(\lambda)} x^A, \quad x^A = x_1^{\#1\text{s in } A} x_2^{\#2\text{s in } A} \dots$$

We can multiply many of the different bases of Λ :

$$e_\lambda e_\mu = e_{\lambda \cup \mu},$$

$$h_\lambda h_\mu = h_{\lambda \cup \mu},$$

$$p_\lambda p_\mu = p_{\lambda \cup \mu}.$$

And multiplying $m_\lambda m_\mu$ is straightforward. What about multiplying Schur functions?

Let $|\mu| + |\nu| = 1$. Then

$$s_\mu s_\nu = \sum_{|\lambda|=n} c_{\mu,\nu}^\lambda s_\lambda$$

What are the coefficients $c_{\mu,\nu}^\lambda$?

Proposition 1.1. $c_{\mu,\nu}^\lambda \in \mathbb{N}$.

Proof. Let S^ν, S^λ be irreducible representations. Then $s_\mu s_\nu$ corresponds to $\text{ind}_{S_k \times S_{n-K}}^{S_n} S^\mu \otimes S^\nu$. So $c_{\mu,\nu}^\lambda$ is the inner product of S^λ with this induced character. This is the dimension of the irreducible representation S^λ in this representation. \square

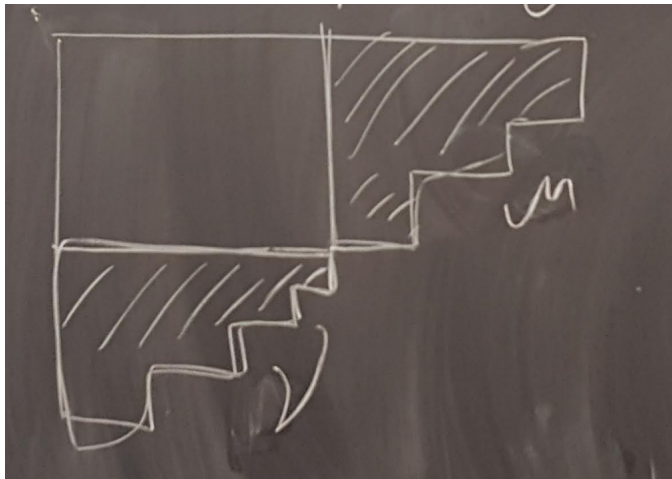
Theorem 1.1. $c_{\mu,\nu}^\lambda = \# \text{LR}(\lambda/\mu, \nu)$, the number of a certain type of semistandard Young tableaux.

This is difficult to prove.¹

¹It is so difficult that Stanley did not actually prove it in his textbook.

1.2 Multiplying Schur functions

Let $\mu \circ \nu$ be the skew shape



Then

$$s_\mu s_\nu = s_{\mu \circ \nu} = \sum_{A \in \text{SSYT}} x^A = \sum_{|\lambda|=n} c_{\mu, \nu}^\lambda s_\lambda$$

How do we determine a tableau with shape $\mu \circ \nu$? Take the skew-shape and reduce it using Jeu-de-taquin.

Example 1.1. We reduce the skew tableau

		1	1
		3	
1	2		
3	3		

to the tableau

1	1	1
2	3	3
3		

So $c_{\mu, \nu}^\lambda$ is the multiplicity of any $P \in \text{SSYT}(\lambda)$ as a jeu-de-taquin of $B \circ C$, where $B \in \text{SSYT}(\mu)$ and $C \in \text{SSYT}(\nu)$.

Corollary 1.1. $c_{\mu, \nu}^\lambda \in \#P$.

There is a polynomial algorithm, jeu-de-taquin, for determining if B and C produce the correct tableau. But this is a very messy combinatorial interpretation. There is a better interpretation.

1.3 Ballot sequences

Definition 1.1. (a_1, \dots, a_n) is a **ballot sequence** if for all $k \in [n]$, the number of i s among a_1, \dots, a_k is greater than the number of $(i+1)$ s among a_1, \dots, a_k for all i .

Example 1.2. The sequence $(1, 1, 2, 1, 1, 2, 3, 3, 1, 2, 3)$ is a ballot sequence.

$\text{Cat}(n)$ is the number of ballot sequences with n 1s and n 2s. Young tableaux are basically the same as ballot sequences; if the number i in our tableau is in row j , we can make the i -th term in the sequence j .

When we have a pair of tableaux that we arrange into a skew shape, form a sequence by listing the numbers in each row from left to right, going down in rows.

Example 1.3.

		1	1
		3	
1	2		
3	3		

gives us the sequence $(1, 1, 3, 3, 1, 3, 3)$.

Theorem 1.2. $c_{\mu, \nu}^{\lambda} = \# \text{SSYT}(\nu, \lambda \setminus \mu)$ such that the sequence obtained from $B \circ C$ is a ballot sequence, where $B \in \text{SSYT}(\mu)$ and $C \in \text{SSYT}(\nu)$.

Next time we will discuss the following.

Corollary 1.2. $c_{\mu, \nu}^{\lambda}$ is the number of integer points in a polytope defined by the vectors λ, μ, ν .

Theorem 1.3. It can be determined in polynomial time whether $c_{\mu, \nu}^{\lambda} = 0$.

Math 206B Lecture 23 Notes

Daniel Raban

March 4, 2019

1 Complexity Aspects of Young Tableau

1.1 Generating random Young tableau

Given λ , we want to generate a random $A \in \text{SYT}(\lambda)$. We have two algorithms for doing this:

1. NPS algorithm: the sort of 2-dimensional bubblesort on tableaux
2. GNW algorithm: Place n according to a hook walk, update λ to be $\lambda \setminus \{n\}$, and repeat.

Which one is faster?

Example 1.1. Let $\lambda = (n)$, which will produce a 1-row tableau. Then NPS will take $\mathbb{E}[\text{inv}(\sigma)] = \Theta(n^2)$ steps. Alternatively, GNW will take $\Theta(n \log(n))$ steps.

Example 1.2. Let λ be the shape of a $k \times k$ square tableau, $\lambda = (k^k)$. Here, $n = k^2$. Then GNW will run in $\Theta(n \log(n))$ time. What about NPS? It goes column by column, and each column takes $\Theta(k^2)$ time. So does NPS run in $\Theta(k^3) = \Theta(n^{3/2})$ time? We really want to look at the average complexity. It turns out that the average complexity is $\Theta(k^3) = \Theta(n^{3/2})$ anyway.

1.2 Generating random partitions

Given n , we want to uniformly generate a random partition $|\lambda| = n$. Here is a (not too) slow algorithm:

Compute $p_k(n)$, the number of partitions λ of n such that $\lambda_1 = k$ parts. Then $p_k(n) = p_1(n - k) + p_n(n - k) + \dots + p_k(n - k)$ and $p_1(n) = 1$, so we can solve the recurrence relation. This takes $\Theta(n^{3.5})$ steps.

What is a smarter algorithm? We know

$$\sum_{n=0}^{\infty} p(n)t^n = \prod_{i=1}^{\infty} \frac{1}{1-t^i}$$

Write this as the following:

$$= \sum_{\lambda \in \mathcal{P}} t^{|\lambda|}.$$

Now think of this probabilistically. Let $Z_i + 1 \sim \text{Geom}(1 - t^i)$. Then let $\lambda = 1^{Z_1} 2^{Z_2} \dots$. Now

$$\mathbb{P}(\lambda) = \mathbb{P}(Z_1 = m_1(\lambda), Z_2 = m_2(\lambda), \dots) = \prod_{i=1}^{\infty} t^{im_i} (1 - t^i) = t^{\sum im_i} \prod_{i=1}^n (1 - t^i).$$

If we generate λ like this, then $|\lambda| = n \iff Z_1 + \dots + 2Z_2 + \dots = n$. So we can generate all these and output λ . So we get

$$\mathbb{P}(|\lambda| = n) = p(n) t^n \prod_{i=1}^n (1 - t^i) \sim e^{c\sqrt{n}} t^n \prod_{i=1}^n (1 - t^i).$$

Optimize over all possible values of t . Then the number of steps becomes $\Theta(n^{3/4})$ or so. This idea is called **Boltzmann sampling**.

1.3 Generating random 3-dimensional partitions

Given n , we want a random 3-dimensional (called solid/plane) partition A . We proved¹ the generating function

$$\sum_{A \in \mathcal{PP}} t^{|A|} = \sum_{n=0}^{\infty} \mathcal{PP}(n) t^n = \prod_{i=1}^{\infty} \frac{1}{(1 - t^i)^i}.$$

We can do the same type of sampling as before. Let $Z_{i,j} \sim \text{Geom}(1 - t^i)$, where $1 \leq j \leq i$. Place the $Z_{i,j}$ in a matrix:

$$\begin{array}{cccc} Z_{1,1} & Z_{2,1} & Z_{3,1} & \cdots \\ Z_{2,2} & Z_{3,2} & \cdots & \\ Z_{3,3} & \cdots & & \end{array}$$

Flip the matrix across the (non-main) diagonal and apply the Hilman-Grassl algorithm.

1.4 Generating random skew shapes

Suppose we have the skew shape $\lambda \setminus \mu$. How do we generate a random $A \in \text{SYT}(\lambda \setminus \mu)$? The Jacobi-Trudy identity gives us

$$s_{\lambda} = \det([h_{\lambda_i - i + j}]).$$

¹Maybe the word ‘proved’ should be taken lightly.

Then we get

$$f^\lambda = n! \det \left(\left[\frac{1}{(\lambda_i - i + j)!} \right] \right)$$

We can calculate the probability of putting n into a corner using

$$\mathbb{P}(n \text{ in corner}) = \frac{f^{\lambda \setminus \{n\}}}{f^\lambda}$$

Here, we use a generalized version of the Jacobi-Trudy identity:

Theorem 1.1 (Jacobi-Trudy). *Let*

$$s_{\lambda \setminus \mu} := \sum_{A \in \text{SSYT}(\lambda \setminus \mu)} t^{|A|}.$$

Then

$$s_{\lambda \setminus \mu} = \det([h_{\lambda_i - \mu_i - i + j}])$$

Math 206B Lecture 24 Notes

Daniel Raban

March 6, 2019

1 Topics on Skew Tableau and Skew Schur Functions

1.1 Complexity of the number of skew tableau

Recall that **skew Schur functions** satisfy

$$s_{\lambda \setminus \mu} = \sum_{A \in \text{SSYT}(\lambda \setminus \mu)} x^A.$$

Let $f^{\lambda \setminus \mu}$ be the number of $\text{SSYT}(\lambda \setminus \mu)$.

Theorem 1.1 (Jacobi-Trudy).

$$f^{\lambda \setminus \mu} = n! \det \left[\left(\frac{1}{(\lambda_i - \mu_i + j - i)!} \right) \right]$$

Corollary 1.1. $f^{\lambda \setminus \mu}$ can be computed in polynomial time.

Theorem 1.2 (Pittner-Pak). *Let D be an arbitrary shape of a diagram (not necessarily a skew shape). Then f^D is #P complete.*

This is equivalent to the following theorem.

Definition 1.1. $\text{Bruhat}(\sigma) = \{\omega \in S_n : \omega \leq \sigma\}$.

Theorem 1.3. $|\text{Bruhat}(\sigma)|$ is #P complete.

1.2 Littlewood-riichardson coefficients for skew Schur functions

Recall that $s_\mu s_\nu = \sum_{|\lambda|=n} c_{\mu,\nu}^\lambda s_\lambda$, where $c_{\mu,\nu}^\lambda$ are the Littlewood-Richardson coefficients.

Theorem 1.4.

$$s_{\lambda \setminus \mu} = \sum_{|\nu|=n-k} c_{\mu,\nu}^\lambda s_\nu.$$

Proof. The idea is to perform jeu-de-taquin on the skew tableau A of shape $\lambda \setminus \nu$. \square

One interpretation of this is that we can use Schur functions to construct $s_{\lambda \setminus \nu}$, which are symmetric functions that end up being nice.

Recall that

$$\begin{aligned} c_{\mu, \nu}^{\lambda} &= \left\langle S^{\lambda}, S^{\mu} \otimes S^{\nu} \uparrow_{S_k \times S_{n-k}}^{S_n} \right\rangle \\ &= \left\langle S^{\lambda} \downarrow_{S_k \times S_{n-k}}^{S_n}, S^{\mu} \otimes S^{\nu} \right\rangle. \end{aligned}$$

by Frobenius reciprocity. Then if π_{λ} is an irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ corresponding to λ and s_{λ} is a character of π_{λ} , then

$$c_{\mu, \nu}^{\lambda} = \langle \pi_{\lambda}, \pi_{\mu} \otimes \pi_{\nu} \rangle.$$

Theorem 1.5 (Knutsen-Tao, c. 2000). *For all k , $c_{\mu, \nu}^{\lambda} > 0$ iff $c_{k\mu, k\nu}^{k\lambda} > 0$.*

Corollary 1.2. *It can be decided if $c_{\mu, \nu}^{\lambda} = 0$ in polynomial time.*

Proof. $c_{\mu, \nu}^{\lambda} \neq 0$ if there exists a rational point in some certain polytope $P(\lambda, \mu, \nu)$ containing $c_{\mu, \nu}^{\lambda}$ integer points. \square

Theorem 1.6. *Computing $c_{\mu, \nu}^{\lambda}$ is $\#P$ complete.*

This is related to the following problem. When does $A + B = C$, when A, B, C are Hermitian matrices?

Theorem 1.7 (Klyacko). *Let A, B, C be hermitian matrices with (vector of) eigenvalues μ, ν, λ , respectively. Then there exist matrices A, B, C solving $A + B = C$ iff $c_{\mu, \nu}^{\lambda} \neq 0$.*

Math 206B Lecture 25 Notes

Daniel Raban

March 8, 2019

1 Inequalities in Algebraic Combinatorics

1.1 Largest number of tableau for a partition of n

Proposition 1.1. $(f^\lambda)^2 \leq n!$

Proof. This is because $\sum_{|\lambda|=n} (f^\lambda)^2 = n!$. □

Here is a restatement of this fact:

Corollary 1.1. $|\text{SYT}(\lambda)|^2 \leq n!$.

This seems much less obvious, and requires RSK to prove it directly.

Corollary 1.2. Denote $D(n) = \max_{|\lambda|=n} f^\lambda$. Then $D(n) \geq \sqrt{n!/p(n)}$, where $p(n)$ is the number of partitions of n .

Here is a conjecture:

Theorem 1.1. The number of $|\lambda| = n$ such that $f^\lambda = D(n)$ is $O(1)$.

This is open, even though it seems like it should be obvious. In fact, we don't know if it is $e^{O(\sqrt{n})}$. The following, however, is known.

Theorem 1.2 (V-K). $D(n) < \sqrt{n!} \alpha^{\sqrt{n}}$ for some $\alpha > 1$.

So we have this upper bound and the lower bound $\sqrt{n!}/\beta^{\sqrt{n}}$. Here is a conjecture that Professor Pak wants to prove:

Theorem 1.3. The following limit exists:

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \left(\frac{D(n)}{\sqrt{n!}} \right).$$

In 1954, someone at Los Alamos, used extra computing power to compute character tables of S_n for $n \leq 15$. They became interested in $D(n)$ and conjectured that $D(n) \leq \sqrt{n!}/n$. This was proven false about 15 years later.

Theorem 1.4 (Bufetov). *Let $H(n) = 1/p(n) \sum_{|\lambda|=n} f^\lambda$. Then the following limit exists:*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \log \left(\frac{H(n)}{\sqrt{n!}} \right).$$

Theorem 1.5 (V-K). *Let $f^\lambda = D(n)$. Then the shape of λ (scaled by \sqrt{n}) looks like a rotated version of the graph*

$$\Phi(x) = \frac{2}{\pi} (x \arcsin(x/\sqrt{2}) + \sqrt{2-x}).$$

Corollary 1.3. *The average longest increasing subsequence of a permutation is $2\sqrt{n}$.*

1.2 Bounds for Littlewood-Richardson coefficients

Theorem 1.6 (PPY). $(c_{\mu,\nu}^\lambda)^2 \leq \binom{n}{k}$.

The upper bound is actually somewhat tight: the idea is to show that

$$\sum_{|\lambda|=n} \sum_{|\mu|=k, |\nu|=n-k} (c_{\mu,\nu}^\lambda)^2 = \sum_{\alpha \in \text{conj}(H=S_k \times S_{n-k})} \frac{z_\alpha(S_n)}{z_\alpha(H)} \alpha \stackrel{\geq}{=} 1 \binom{n}{k}.$$

Then

$$\max_{\lambda,\mu,\nu} c_{\mu,\nu}^\lambda \geq \frac{\sqrt{\binom{n}{k}}}{\sqrt{p(k)p(n-k)p(n)}}.$$

Proof. The idea of the proof of the theorem is to show that $\binom{n}{k} f^\mu f^\nu$ is the dimension of $\text{ind}_{S_k \times S_{n-k}}^{S_n} S^\mu \otimes S^\nu$ and decompose the representation into irreducible representations. Then

$$\sum_{|\mu|=k} \sum_{|\nu|=n-k} c_{\mu,\nu}^\lambda f^\mu f^\nu = f^\lambda,$$

and

$$\sum_{|\lambda|=n} (c_{\mu,\nu}^\lambda)^2 \leq \sum_{|\lambda|=n} c_{\mu,\nu}^\lambda \frac{f^\lambda}{f^\mu f^\nu} = \frac{1}{f^\mu f^\nu} f^\mu f^\nu \binom{n}{k} = \binom{n}{k}.$$

So $(c_{\mu,\nu}^\lambda)^2 \leq \binom{n}{k}$. □

Theorem 1.7 (PPY, 2018). *There exist λ, μ, ν such that $c_{\mu,\nu}^\lambda = 2^n / e^{O(-\sqrt{n})}$.*

1.3 Bounds on the number of skew tableau of size n

Let $f^{\lambda \setminus \mu} = |\text{SYT}(\lambda \setminus \mu)|$. We know that this number is the determinant of a matrix we get from λ, μ . Can we understand this number better? Our previous considerations give us the following:

Proposition 1.2. *Let $|\lambda| = n$ and $|\mu| = k$. Then*

$$f^{\lambda \setminus \mu} \leq \sqrt{\binom{n}{k} p(n-k) \sqrt{(n-k)!}}.$$

Proof. This follows from the previous inequalities applied to the identity:

$$f^{\lambda \setminus \mu} = \sum_{|\nu|=n-k} c_{\mu, \nu}^{\lambda} f^{\nu}. \quad \square$$

What about lower bounds?

Theorem 1.8 (Naruse, MPP).

$$f^{\lambda \setminus \mu} = n! \sum_{D \in \mathcal{E}(\lambda \setminus \mu)} \prod_{(i,j) \notin D} \frac{1}{h_{i,j}},$$

where $\mathcal{E}(\lambda \setminus \mu)$ is the set of “excited diagrams” (start with chips in the removed shape μ , and move them to the right or down to get a configuration in $\lambda \setminus \mu$).

Example 1.1. Suppose $\lambda = (3, 3)$ and $\mu = (2)$. Then we start with

★	★	

We get 3 excited diagrams:

★	★	

★		
		★

	★	★

If we take only the first term of the sum, we get the following lower bound:

Corollary 1.4.

$$f^{\lambda \setminus \mu} \geq n! \prod_{(i,j) \in \lambda \setminus \mu} \frac{1}{h_{i,j}}$$

Math 206B Lecture 26 Notes

Daniel Raban

March 11, 2019

1 D -Finite and Polynomially Recursive Series

Note: Today's lecture is a guest lecture. The lecture material is from section 6.4 of Stanley's Enumerative Combinatorics (Volume 2).

1.1 D -finite series

Let K be a field. We have the ring of formal power series $K[[x]]$.

Definition 1.1. If $u = \sum_n f(n)x^n$, then the **formal derivative** is $\frac{d}{dx}u = u' = \sum_n n f(n)x^{n-1}$.

Proposition 1.1. Let $u \in K[[x]]$. The following are equivalent.

1. $\dim_{K(x)}(K(x)u + K(x)u' + K(x)u'' + \cdots) < \infty$.

2. There are $p_0, \dots, p_\ell \in K[x]$ with $p_\ell \neq 0$ such that

$$p_d u^{(d)} + p_{d-1} u^{(d-1)} + \cdots + p_1 u' + p_0 u = 0.$$

3. There are $q_0, \dots, q_m, q \in K[x]$ with $q_m \neq 0$ such that

$$q_m u^{(m)} + q_{m-1} u^{(m-1)} + \cdots + q_1 u' + q_0 u = q.$$

Proof. (1) \implies (2): Suppose $\dim = d$. Then $u, u', \dots, u^{(d)}$ are linearly dependent over $K(x)$. Write down the dependence relation, and clear the denominators to get the p_i .

(2) \implies (3): This is a special case.

(3) \implies (2): Suppose that $\deg_x(q(x)) = t \geq 0$. Differentiate the polynomial relation $t+1$ times to get a homogeneous relation involving the derivatives of u . We get $p_d = q_m \neq 0$. Solve for $u^{(d)}$ to get

$$u^{(d)} \in K(x)u + K(x)u' + \cdots + K(x)u^{(d-1)}.$$

Writing $u^{(d)}$ as a linear combination of $u, \dots, u^{(d-1)}$ and differentiating gives $u^{(d+1)} \in K(x)u + \cdots + K(x)u^{(d)} = K(x)u + \cdots + K(x)u^{(d-1)}$. By induction, for all $k \geq 0$, $u^{(d+k)} \in K(x)u + \cdots + K(x)u^{(d-1)}$. \square

This allows us to make the following definition.

Definition 1.2. $u \in K[[x]]$ is *D-finite* if any of these three conditions hold for u .

Example 1.1. $u = e^x = \sum_n x^n/n!$ is *D-finite*. This satisfies $u' = u$, so $u' - u = 0$. In general, $u = x^m e^{ax}$ is *D-finite*, as $u' = mx^{m-1}e^{ax} + ax^m e^{ax} = (m/x + a)u$.

Example 1.2. Suppose $u = \sum_{n \geq 0} n!x^n$. Then $(xu)' = \sum_{n \geq 0} (n+1)!x^n$, so we see that $1 + x(xu)' = u$. That is, $x^2u' + (x-1)u = -1$, so u is *D-finite*.

1.2 Polynomially recursive series

Definition 1.3. Say $f : \mathbb{N} \rightarrow K$ is *P-recursive* (or **polynomially recursive**) if there are $P_0, \dots, P_e \in K[x]$ with $P_e \neq 0$ such that

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \dots + P_0(n)f(n) = 0$$

for all $n \in \mathbb{N}$.

Proposition 1.2. Let $u = \sum_{n \geq 0} f(n)x^n \in K[[x]]$. Then u is *D-finite* if and only if f is *P-recursive*.

Proof. First, suppose u is *D-finite*. Then we have polynomials p_I such that

$$p_e u^{(d)} + \dots + p_1 u' + p_0 u = 0.$$

Then $x^j u^{(i)} = \sum_{n \geq 0} (u+i-j)_i f(u+i-j)x^n$, where $(u+i-j)_i$ is the falling factorial. For $k \gg 0$, equate coefficients of x^{n+k} in the polynomial relation to get the recurrence. Note that if $[x^j]p_d(x) \neq 0$, then $[n^d]P_{d-j+k}(n) \neq 0$ (where brackets denote the coefficient of the term inside).

Now suppose f is *P-recursive*. Then f satisfies a relation

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \dots + P_0(n)f(n) = 0$$

with $P_e \neq 0$. For each i , $\{(n+i)_j : j \geq 0\}$ is a basis for the K -vector space $K[u]$. So $P_i(n)$ is a K -linear combination of $(n+i)_j$ s. So $\sum_n P_i(n)f(n+i)x^n$ is a K -linear combination of series of the form $\sum_{n \geq 0} (n+i)_j f(n+i)x^n$. Now $\sum_{n \geq 0} (n+i)_j f(n+i)x^n = R_i(x) + x^{j-i}u^{(j)}$, where $R_i \in x^{-1}K[x^{-1}]$. For example, $x^{-1}u' = \sum_{n \geq -1} (n+2)f(n+2)x^n = f(1)x^{-1} + \sum_{n \geq 0} (n+2)f(n+2)x^n$. Now multiply the relation by x^n and sum over $n \geq 0$ to get

$$0 = \sum a_{i,j} x^{j-i} u^{(j)} + R(x).$$

This sum is finite, $a_{i,j} \in K$ are not all 0, and $R(x) \in x^{-1}K[x^{-1}]$. Multiply by x^q with $q \gg 0$. We get that u is *D-finite*. \square

Example 1.3. Let $u = e^x$. Another way to show that u is D -finite is to show that $f(n) = n!$ is P -recursive: $f(n+1) - (n+1)f(n) = 0$.

Example 1.4. Let $f(n) = \binom{2n}{n}$. This is P -recursive: $(n+1)f(n+1) - 2(2n+1)f(n) = 0$. So $u = \sum_{n \geq 0} \binom{2n}{n} x^n$ is D -finite. This is the series for $1/\sqrt{1-4x}$.

Proposition 1.3. Suppose $f, g : \mathbb{N} \rightarrow K$ is P -recursive and $f(n) = g(n)$ for all sufficiently large n . Then g is P -recursive.

Proof. Suppose $f(n) = g(n)$ for $n \geq n_0$. Then

$$\left(\prod_{j=0}^{n_0-1} (n-j) \right) [P_e(n)g(n+e) + \cdots + P_0(n)g(n)] = 0,$$

so g is P -recursive. □

Theorem 1.1. Suppose $u \in K[[x]]$ is algebraic over $K(x)$ of degree d . Then u is D -finite and satisfies a polynomial equation of order d .

Proof. Let $P(Y) \in K(x)[Y]$ be the minimal polynomial of u over $K(x)$. Suppose $P(Y) = \sum_{i=0}^d p_i Y^i$. Then $P(u) = 0$. Differentiate to get

$$0 = (P(u))' = \left(\sum_{i=0}^d p_i u^i \right)' = \underbrace{\sum_{i=0}^d p'_i u^i}_{=Q(u)} + \underbrace{\sum_{i=0}^d i p_i u^{i-1} u'}_{=(\frac{\partial P}{\partial Y}(u))u'}$$

The derivative $\frac{\partial P}{\partial Y}(u) \neq 0$. So $u' = Q(u)/(\frac{\partial P}{\partial Y}(u)) \in K(x)(u)$. Similarly, $u^{(n)} \in K(x)(u)$ for all N . Then we get linear dependence among $u, u', \dots, u^{(d)}$. □

Math 206B Lecture 27 Notes

Daniel Raban

March 13, 2019

1 Operations on D -Finite Series

Note: Today's lecture is a guest lecture.

1.1 Addition, multiplication, and composition of D -finite series

Last time, we showed that $u \in K[[x]]$ is D -finite iff $\dim_{K(x)}(\text{span}(\{u, u', u'', \dots\})) < \infty$.

Theorem 1.1. *The set D of D -finite $u \in K[[x]]$ is a subalgebra of $K[[x]]$. If $u, v \in D$ and $\alpha, \beta \in K$, then $\alpha u + \beta v \in D$, and $uv \in D$.*

Proof. Given $w \in K[[x]]$, let $V_w = \text{span}_{K(x)}(\{w, w', w'', \dots\}) \subseteq K((x))$. Suppose $u, v \in D$, $\alpha, \beta \in K$, and let $y = \alpha u + \beta v$. Then $y, y', y'', \dots \in V_u + V_v$. Thus, $\dim(V_y) \leq \dim(V_u + V_v) \leq \dim(V_u) + \dim(V_v) < \infty$.

Next, let $u, v \in D$. Consider $\phi : V_u \otimes_{K(x)} V_v \rightarrow K((x))$ defined by $\phi(y \otimes z) = yz$ for all $y \in V_u$ and $z \in V_v$. The product rule implies $V_{uv} \subseteq \phi(V_u \otimes_{K(x)} V_v)$; indeed, $(uv)^{(i)} = \sum_{j=0}^i \binom{i}{j} u^{(j)} v^{(i-j)}$. Thus, $\dim(V_{uv}) \leq \dim(V_u \otimes_{K(x)} V_v) = \dim(V_u) \dim(V_v) < \infty$, so $uv \in D$. \square

Theorem 1.2. *Let $u \in D$ and $v \in K_{\text{alg}}[[x]]$ (i.e. $v \in K[[x]]$ and v is algebraic over $K(x)$) with $v(0) = 0$. Then $u(v(x)) \in D$.*

Proof. Let $y = u(v(x))$. Then $y' = u'(v(x))v'(x)$, $(u'(v(x)))' = u''(v(x))v'(x)$, etc. In general, $y^{(i)}$ is a linear combination of $u(v(x)), u'(v(x)), u''(v(x)), \dots$ with coefficients in $K[v, v', v'', \dots]$. Since v is algebraic over $K(x)$, $v^{(i)} \in K(x, v)$ for all i (proved last time). Thus, $K[v, v', \dots] \subseteq K(x, v)$.

Let $V = \text{span}_{K(x)}(\{u(v(x)), u'(v(x)), \dots\}) \ni y^{(i)}$ for all i . We want to show that $\dim_{K(x)}(V) < \infty$. Since u is D -finite, $\dim_{K(x)}(\text{span}_{K(x)}(\{u(x), u'(x), \dots\})) < \infty$. By “specializing x at v ,” $\dim_{K(x)}(\text{span}_{K(x)}(\{u(v(x)), u'(v(x)), \dots\})) < \infty$. So we get that $\dim_{K(x, v)}(\text{span}_{K(x, v)}(\{u(v(x)), u'(v(x)), \dots\})) < \infty$. Then $\dim_{K(x, v)}(V) < \infty$, and (since v is algebraic over $K(x)$) $[K(x, v) : K(x)] < \infty$, so

$$\dim_{K(x)}(V) = (\dim_{K(x, v)}(V)) \cdot [K(x, v) : K(x)] < \infty. \quad \square$$

Example 1.1. We know that $\sum_{n \geq 0} n!x^n$, e^x , and $\frac{x}{\sqrt{1-4x}}$ are in D . So we can get that $u = (\sum_{n \geq 0} n!x^n)e^{x/\sqrt{1-4x}} \in D$. This would be difficult to do by hand without the results we have proved.

1.2 Hadamard products of P -recursive series

Given $h : \mathbb{N} \rightarrow K$ and $R(n) \in K(n)$, we want to define $Rh : \mathbb{N} \rightarrow K$ by $Rh(n) = R(n)h(n)$. But this could be undefined when $R(n) = \infty$. Here is the solution: given $h_1, h_2 : \mathbb{N} \rightarrow K$, say $h_1 \sim h_2$ if $h_1(n) = h_2(n)$ for all $n \gg 0$. Call $[h]$, the equivalence class of h , the **germ** of h . Define $\mathcal{G} = \{[h] \mid h : \mathbb{N} \rightarrow K\}$; this is a $\mathbb{K}(n)$ vector space. Note: if $g \sim h$, the nh is P -recursive iff g is P -recursive. For each $h : \mathbb{N} \rightarrow K$, set $\mathcal{G}_h = \text{span}_{K(n)}(\{[h(n)], [h(n+1)], [h(n+2)], \dots\})$.

Fact: h is P -recursive iff $\dim_{K(n)} \mathcal{G}_h < \infty$.

Definition 1.1. Given $u = \sum_n f(n)x^n$ and $v = \sum_n g(n)x^n$, define the **Hadamard product** $u * v = \sum_{n \geq 0} f(n)g(n)x^n$.

Theorem 1.3. If $f, g : \mathbb{N} \rightarrow K$ are P -recursive, then so is fg . That is, $u, v \in D \implies u * v \in D$.

Proof. It is sufficient to show that if $[f], [g]$ are P -recursive (element member of the germ is recursive), then so is $[fg]$. Define $\phi : \mathcal{G}_f \otimes_{K(n)} \mathcal{G}_g \rightarrow \mathcal{G}$ such that for each i, j , on simple tensors, $\phi([f(n+i)] \otimes [g(n+j)]) = [f(n+i)][g(n+j)] = [f(n+i)g(n+j)]$. So the image of ϕ contains $\mathcal{G}_{fg} = \text{span}_{K(n)}(\{[f(n)g(n)], [f(n+1)g(n+1)], \dots\})$. So

$$\dim_{K(n)}(\mathcal{G}_{fg}) \leq \dim_{K(n)}(\mathcal{G}_f \otimes \mathcal{G}_g) = (\dim(\mathcal{G}_f))(\dim(\mathcal{G}_g)) < \infty.$$

So fg is P -recursive. □

1.3 Fun facts

Here are some fun facts from Professor Pak's 2016 206A notes:

Theorem 1.4. Let $S \subseteq \mathbb{Z}^d$ with $|S| < \infty$. Let a_n be the number of walks $0 \rightarrow 0$ of length n on \mathbb{Z}^d with steps in S . Then (a_n) is P -recursive.

Theorem 1.5. If $a_n = |\{\sigma \in S_n : \sigma^2 = 1\}|$, then $a_n = a_{n-1} + (n-1)a_{n-2}$, so (a_n) is P -recursive.

Definition 1.2. Given $F = \sum f(n_1, n_2, \dots, n_r)x_1^{n_1} \cdots x_r^{n_r} \in F[[a_1, \dots, x_r]]$, define the **diagonal**, $\text{diag}(F) \in K[[t]]$, by

$$(\text{diag}(F))(t) = \sum_n f(n, n, \dots, n)t^n.$$

Theorem 1.6 (Furstenberg). *Suppose $F(s, t) \in K[[s, t]] \cap K(s, t)$. Then $\text{diag}(F)$ is algebraic. If $P, Q \in \mathbb{Z}[x_1, \dots, x_r]$, then $\text{diag}(P/Q)$ is D -finite.*

The proof of this theorem involves Puiseux series.

Remark 1.1. The converse is also true. An algebraic single variable power series is the diagonal of such a multivariable series.