1 Graphs of convex polytopes

Reminder: simple polytopes are those whose vertices have degree d.

Theorem 1.1 (Blind, Mani 197x)**.** *Let $P \subset \mathbb{R}^d$ be simple. Then the face-lattice of P is determined by $G(P)$.*

All faces are set, the moment you decide the vertices and edges. "This took decades to figure out."

1.1 Why you should not believe in this type of result

There exists polytopes P, Q such that $G(P) \simeq G(Q)$, but $P \not\simeq Q$. If $P = \Delta^5$, then $G(P) = K_6$. For Q, put two triangles on orthogonal planes in $\mathbb{R}^4$. We claim that $G(Q) = K_6$ as well.

There's a whole class of polytopes, that of *neighborly* polytopes, all of whom satisfy $G(P) = K_n$.

Theorem 1.2. *For general polytopes P, all faces of* $\dim \leq d/2$ *are needed to uniquely determine the face-lattice of P (necessary and sufficient).*

Proof by Kalai of Theorem 1.1. Another use of Morse functions. Let φ be Morse and consider the orientation $\mathcal{O}_\varphi$ induced on P by φ. Once more let h_i be the number of vertices with out-degree i (we proved last time that these are independent to our choice of Morse function). Then the number of faces of P is

$$h_0 + 2h_1 + 4h_2 + \cdots + 2^d h_d \tag{1}$$

(again, this follows from our proof of Dehn-Sommerville on Monday).

Let $\mathcal{O}$ be an arbitrary *acyclic orientation* of $G(P)$. We can still define $h_i^{\mathcal{O}}$ in the same way as above. Let

$$\alpha(\mathcal{O}) = h_0^{\mathcal{O}} + 2h_1^{\mathcal{O}} + \cdots + 2^d h_d^{\mathcal{O}}. \tag{2}$$

Lemma 1.3. *$\alpha(\mathcal{O}) \geq |\mathcal{F}|$, the total number of faces of P.*

Proof of lemma. Easy: recall our bijective proof from last time based on minimal vertices. Here we no longer have a bijection, but the idea still carries through. ☐

We say that $\mathcal{O}$ is *good* if $\alpha(\mathcal{O}) = |\mathcal{F}|$ (there do exist a.o.'s that do not correspond to Morse functions).

Definition 1. $X \subset V(G)$ is *final* if there is a (any) good acyclic orientation $\mathcal{O}$ of G so that $H = G|_X$ is k-regular and has no out-edges.

Lemma 1.4. *Faces in P correspond directly to final subsets $X \subset V(G)$.*

Note: this gives us an algorithm (useless in practice) for finding all faces of P.

Proof of lemma. Let F be a face of P. We prove that $V(F)$ is a final subset. Let H be any supporting hyperplane of F, and perturb H into H' Define φ by letting $\varphi(H') = 0$ and increasing as we move away from the polytope. This gives us an a.o. which makes $V(F)$ final. Regularity is ensured by the fact that P is simple.

Conversely, suppose that X is a final subset with respect to a good a.o. $\mathcal{O}$. We prove that X is a face in P. In $\mathcal{O}|_X$ there must be at least one *source* vertex $z \in X$. We are going to use goodness to show that z is unique. Let $H = G|_X$ and let $H' \subset G$ be the vertices of a face F with source at z (we can do this: take all the outgoing edges of z as our subspace generating set). Since X is final, $H' \subset H$, as we can never leave X. Finally, since H' is a face, we know that it's k-regular.

"And now a fantastic sentence:" if a connected k-regular graph sits inside another connected k-regular graph, we in fact have equality. Thus, $H = H'$. $\square$

$\square$

1 Graphs of polytopes

Theorem 1.1 (Balinski, 1961). *Let $P \subset \mathbb{R}^d$ be a convex polytope with $\dim P = d$. Then $G(P)$ is d-connected.*

1961 is around the time that linear programming, simplex method type stuff started up. Also, note that the theorem is obvious in dimensions 2 and 3.

Definition 1. A graph G is *d-connected* if after removing any $d-1$ vertices from G we still have a connected graph.

Proof. Let $X \subset V = V(G(P))$ be a set of $d-1$ vertices, $X = \{x_1, \ldots, x_{d-1}\}$. Fix a vertex $z \in V \setminus X$. Denote by H the hyperplane spanned by $X \cup \{z\}$ (d points in d-dimensional space create a hyperplane)(we can of course always find z to ensure that H is unique, but I'm not sure we care).

Let ψ be a nonzero linear function with $\psi(H) = 0$, and let φ be a Morse function obtained by a small perturbation of ψ. Orient all edges as in Figure . Now, the top v and bottom w of the polytope form sinks.

Lemma 1.2. *Every vertex y with $\varphi(y) > 0$ is connected to v, which maximizes φ on P.*

Proof of Lemma. Suppose y does not maximize φ. Then there are vertices above it and connected to it which are bigger, so move to those. Eventually this process must terminate at v. □

To finish, we observe that z is connected to v and w. Thus, the graph is connected. □

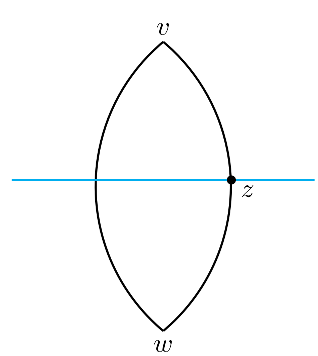

Figure 1:

2 Examples of polytopes

1. The d-simplex in $\mathbb{R}^d$ is the convex hull of $d+1$ points in general position. It is the only polytope that is both simple *and* simplicial. Additionally, it

is self-dual, and

$$f_i = \binom{d+1}{i+1}; \quad h_i = 1. \tag{1}$$

2. The d-cube. We already calculated that

$$f_i = 2^{d-i}\binom{d}{i}; \quad h_i = \binom{d}{i}. \tag{2}$$

2′. The d-cross polytope is the dual to the d-cube (generalizes the octahedron).

3. The *Birkhoff polytope* B_n is the convex hull of all $n \times n$ bistochastic matrices $M = (m_{ij})$, with $m_{ij} \geq 0$. First, observe that $\dim B_n = (n-1)^2$, as the matrix is determined by the first $(n-1) \times (n-1)$ minor. Second, observe that the vertices of B_n are the permutation matrices M_σ, with $m_{ij} = 1$ if and only if $\sigma(i) = j$. Thus, $|V| = n!$. The number of facets f_{d-1} with $d = (n-1)^2$ is equal to n^2, as each facet is given by $m_{i,j} \geq 0$. B_n is neither simple nor simplicial: each vertex M_σ belongs to $n^2 - n$ facets, but $d = n^2 - 2n + 1$.

   It's clear that each M_σ is a vertex, since they are clearly extreme points. To argue that they are the only vertices, just show that everything is a convex combination of the M_σ.

   In general, f_i is complicated (though there is a description). f_1 is nice enough to present today, however. In fact, we claim that (M_σ, M_ω) is an edge in B_n if and only if $\sigma\omega^{-1}$ is a cycle in S_n.

   To see this, let's suppose that $\omega = (1)$ and that σ is a product of two **disjoint** cycles $c_1 c_2$ (we do allow additional fixed points, so cycles of length > 1). Consider $A = \alpha M_\sigma + (1-\alpha)I$. But then if $\sigma' = c_1, \sigma'' = c_2$, we can construct $D = \alpha M_{\sigma'} + (1-\alpha)I$, $E = \alpha M_{\sigma''} + (1-\alpha)I$. Observe that $A = \frac{1}{2}(D+E)$, so there are two different affine combinations of vertices for A. Thus, A cannot lie on an edge. The other direction, is easier.

   We now claim that

$$f_1 = \frac{\deg(n) \cdot n!}{2}, \tag{3}$$

   where $\deg(n)$ is equal to the number of cycles in S_n. Last quarter we established that

$$\deg(n) = \sum_{\ell=1}^{n} \binom{n}{\ell}(\ell-1)! = n! \sum \frac{1}{(n-\ell)!\ell} = \theta((n-1)!) \tag{4}$$

   ("the total number of cycles is not much more than the number of n-cycles").

1 Birkhoff polytope

Proposition 1.1. *Let B_n be the Birkhoff polytope. Then the diameter of $G(B_n)$ is equal to 2. This is equivalent to saying that every $\sigma \in S_n$ can be written as $\sigma = \omega_1\omega_2$, where ω_i are cycles.*

No proof: refer to last time where we showed an edge criterion.

In fact, we have a stronger result.

Theorem 1.2 (P.)**.** *The mixing time of $G(B_n)$ is equal to 2. Equivalently,*

$$||Q^2 - U||_{TV} \to 0 \tag{1}$$

as $n \to \infty$, where $|| - ||_{TV}$ is the total variation distance, U is the uniform distribution and Q^2 is a random edge-walk.

Proposition 1.3. *B_n can be inscribed (topologically) into S^d with $d = (n-1)^2$.*

Proof. Let O be the matrix with every entry $= 1/n$. Then

$$|M_\sigma - O| = \left[(n^2 - n)\frac{1}{n^2} + n\left(1 - \frac{1}{n}\right)^2\right]^{1/2}, \tag{2}$$

which simplifies to $\sqrt{n-1}$. $\square$

2 Permutohedron

We now introduce a different polytope with $n!$ vertices. For $\sigma \in S_n$, let $a(\sigma) = (\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n$.

Definition 1. The *permutohedron* P_n is defined as the convex hull,

$$P_n = conv\{a(\sigma) : \sigma \in S_n\}. \tag{3}$$

Since $\sum \sigma(i) = \binom{n+1}{2}$, we observe that $\dim P_n = n - 1$. Additionally, we can observe that P_n is inscribed into S^{n-1}, due to the fact that

$$\sum_{i=1}^{n} \sigma(i)^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4}$$

As a consequence of these two observations:

Proposition 2.1. *P_n has $n!$ vertices.*

Theorem 2.2. *$G(P_n) = Cayley(S_n, \{(12), (23), \ldots, (n-1n)\})$.*

Corollary 2.3. *P_n is simple.*

Proof of Theorem 2.2. Observe that P_n is S_n-invariant. S_n acts on $\mathbb{R}^n$ by permuting coordinates, so it suffices to prove the theorem for one vertex, specifically the vertex with $\sigma = 1, 2, \ldots, n$. Let $a(\sigma) = (1, 2, \ldots, n)$ and suppose we are adjacent to $a(\omega)$. We'd like to show that $a(\omega) = (1, 2, \ldots, i+1, i, \ldots, n)$ for some i.

The difference vector is equal to $(0, \ldots, 1, -1, \ldots, 0)$. Notice that we can arrive at any permutation using such difference vectors (a *positive* combination, which is necessary for our next sentence). So, every permutation lies in a cone spanned by the possible difference vectors. Thus the only possible edges are the edges of the cone, which are exactly the permutation vectors of the desired form. Full dimensionality implies that we in fact have every such edge. □

This is slightly misleading: the adjacencies for non-identity permutations still swap i and $i+1$ instead of the coordinates. For example, 24135 is adjacent to $14235, 34125, 23145$ and 25134.

Our knowledge of the Cayley graph immediately gives us a diameter of $\binom{n}{2}$ (an observation) and a mixing time of $\theta(n^3 \log n)$ (a hard theorem).

2.1 The f-vector of the permutohedron

Let $f_i = f_i(P_n)$ First, $f_{n-1} = 1$.

$$f_{n-2} = ? \tag{5}$$

Let's look at the h-vector instead. Consider the function

$$\varphi = x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots + \varepsilon^{n-1} x_n, \tag{6}$$

where ε is really small. This is a Morse function, so let's try and figure out the out-degree of $a(\sigma)$. An example: consider $\sigma = 24135$. Switching to 14235 lowers the value of φ, while switching to 34125 raises it. In general, the out-degree is the number of *descents* of σ. Thus,

$$h_k = \#\{\sigma \in S_n : des(\sigma) = k\} = A(n, k), \tag{7}$$

an Euler number. Recall the easy recurrence,

$$A(n, k) = (n-k)A(n-1, k-1) + (k+1)A(n-1, k). \tag{8}$$

Now, how can we get to the f-vector? For $n = 3$, hexagon. For $n = 4$, truncated octahedron, with 8 hexagons and 6 squares. Non-adjacent transpositions form a square, while adjacent ones form a hexagon (copy of S_3).

1 Permutohedron

I was late! $P_n \subset \mathbb{R}^n$ is defined by

$$P_n = conv(a(\sigma : \sigma \in S_n)) \tag{1}$$

where $a(\sigma) = (\sigma(1), \ldots, \sigma(n))$.

Theorem 1.1. *Permutohedron is simple, $n-1$ dimensional and $h_k = A(n,k)$*

Theorem 1.2. *The graph of the Permutohedron is the Cayley graph.*

Anyway, Igor estalishes that

$$f_k = k!S(n,k) \tag{2}$$

via a bijection(?) to cosets(?) of S_n.

2 Associahedron

Let's recap on the number of vertices:

- Simplex: n
- Cube: 2^n
- Permutohedron: $n!$
- Associahedron: C_n, the n-th Catalan number.

Igor shall use Q_n to refer to the associahedron. Recall of course that

$$C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{3}$$

Definition 2.1. Q_n is defined by letting V be all triangulations of an $n+2$-gon W_{n+2}, and letting the faces be all subdivisions of W_{n+2} with non-intersecting diagonals.

Example: for $n = 2$, we get S^0. For $n = 3$, we get a pentagon (see Igor's book, chapter 8 for a good picture).

Theorem 2.2 (Stasheff conjectured in 1963, Lee proved in 1989)**.** *Q_n is geometrically realizable as an $(n-1)$-dimensional polytope. That is, there is a polytope of dimension $n-1$ whose face lattice matches the one described above.*

2.1 Edges

The edges of Q_n correspond to pairs of triangulations, from which one can reach the other by a "flip" (again, Igor's book chapter 8 for a good picture).

Observation 2.3. Q_n is simple.

This is based off the fact that the number of diagonals off one fixed vertex is $n-1$ (?).

3 The GZK-construction ($\approx$ 1990)

Let $W = W_{n+2}$ be a convex polygon with vertices $1, \ldots, n+2$. For a triangulation T of W, let $f_T : [n+2] \to \mathbb{R}$ be the function given by

$$f(i) = \sum_{\Delta \ni i} area(\Delta). \tag{4}$$

The set of such functions can be viewed as a point in $\mathbb{R}^{n+2}$. Then let

$$Q_n = conv\{f_T\} \subset \mathbb{R}^{n+2}. \tag{5}$$

Theorem 3.1 (Gelfand-Zelevinsky-Kapranov)**.** *Q_n has the Stasheff face structure. That is, this is in fact a geometric realization of the associahedron.*

4 Another realization

Igor provided another, more combinatorial realization of the associahedron, by Loday (2004). Consider the set of full binary trees, a binary tree where every vertex has either 2 or 0 children (note that these are a Catalan object).

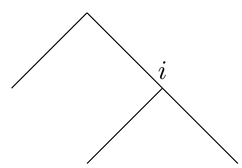

Figure 1: A full binary tree

For a given tree, let f be a function from the non-leaf vertices to $\mathbb{R}$, defined by

$$f(i) = a_i b_i, \tag{6}$$

where a_i is the number of leaves to the left of i, and b_i the number to the right. Additionally, we label our vertices in depth-first search order. For example in Figure 1, $a_i = b_i = 1$.

Now for $\mathcal{T}_n$ the set of full binary trees with n non-leaf vertices ($2n+1$ vertices total), we have the functions f_T for T a particular tree. Again, we think of each f_T as a point in $\mathbb{R}^n$.

Theorem 4.1 (Loday, 2004)**.** *The polytope*

$$Q_n = conv(f_T : T \in \mathcal{T}_n) \tag{7}$$

has the Stasheff face structure, and hence is another geometric realization of the associahedron.

Initally Igor thought these two realizations were the same, but they aren't. The GZK-construction can be parametrized into a family (slightly bending the original polygon changes the area of the triangles), but "somehow the Loday construction is unique."

1 Associahedron continued

Recall:

Theorem 1.1 (Loday)**.** *Let $\mathcal{T}_n$ be the set of full binary trees with $n+1$ leaves, and let*

$$Q_n = conv\{a(t) : t \in \mathcal{T}_n\}. \tag{1}$$

Then Q_n is a Stasheff polytope, i.e. it is a realization of the associahedron.

Some clarification: Stasheff introduced the lattice structure of the associahedron, but only conjectured that it could be realized as a polytope. Igor is using "Stasheff" and "associahedron" somewhat interchangeably. The name "associahedron" is also kind of silly: it's coming from the "balanced parentheses" Catalan structure.

1.1 Example (Loday)

Let $n = 3$.

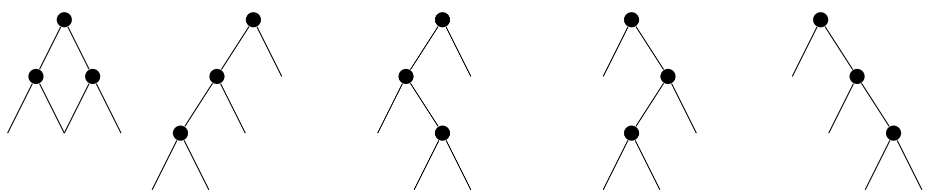

Figure 1: The five full binary trees with 3 interior vertices

The trees in Figure 1 yield the $(a_i b_i)$-vectors (in-order traversal for the coordinates) $(1,4,1),(1,2,3),(2,1,3),(3,1,2),(3,2,1)$.

We know that Q_3 has dimension 2, and we notice that all these points lie on the hyperplane $x + y + z = 6$.

Claim 1.2. *In general, the hyperplane is $\mathcal{H} = \{x_1 + \cdots + x_n = \binom{n+1}{2}\}$.*

Igor drew a projection onto the first two axes and pointed out that two pairs of edges were each parallel. A high degree of paralellism is a special property of Loday's construction.

1.2 Example (GZK)

Once more let $n = 3$. We take the regular pentagon as our initial object to triangulate: the area of the central triangle will be β, and the other two have area α. The coordinates are therefore going to become

$$(2\alpha + \beta, \alpha, \alpha + \beta, \alpha + \beta, \alpha) \tag{2}$$

and its cyclic shifts. Note that this cyclicity means it's a regular pentagon, and thus has no parallel edges.

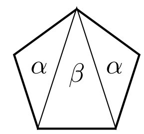

Figure 2: I hate TikZ SO MUCH

1.3 Proof of claim

A proof by picture: we start with a full binary tree with $n+1$ leaves, and consider the left and right subtrees.

Figure 3: Left and right subtrees

Claim 1.3. *The associahedron Q_n is equal to the intersection $\mathcal{H} \cap \mathcal{H}_J$, where*

$$J = \{k, k+1, \dots, \ell\} \tag{3}$$

$$\mathcal{H}_J = \left\{ x_k + x_{k+1} + \dots + x_{k+\ell} \geq \binom{\ell - k + 2}{2} \right\} \tag{4}$$

Again, the proof is by picture:

Figure 4: uhh

Together, these two claims give Loday's theorem.

1.4 Another property

Theorem 1.4. *Q_n is simple in $\mathbb{R}^{n-1}$, and*

$$h_i = \frac{1}{n}\binom{n}{i}\binom{n}{i+1} = N(n,i), \tag{5}$$

the Narayana numbers.

One can interpret the Narayana numbers in many combinatorial ways, but the one most useful to us is the number of (not full) binary trees with n wertices, and k left edges. In particular, we draw special attention to the fact that $N(n,k)$ sum up to C_n, and thus

$$h_0 + h_1 + \dots + h_{n-1} = f_0 = C_n. \tag{6}$$

We can prove that $N(n,k)$ has this interpretation by induction, or some bijective stuff.

Proof using GZK. (Can be done using Loday as well) Start with the standard bijection between triangulations and binary trees. Introduce a Morse function:

$$\varphi = x_1 + \varepsilon x_2 + \varepsilon^2 x_3 + \cdots \tag{7}$$

where ε is some tiny positive number.

We consider what happens under a flip: Note that the flip increases the

Figure 5: flip

area at vertex 1, and so this flip is "better" according to our Morse function. This gives us an orientation on the edges, which correspond to flips. Since the increasing flips turn a left edge into a right edge... something something Narayana numbers. □

1 Rational and irrational polytopes

Definition 1.1. Today we consider a polytope to be defined by its *face structure*, the lattice of faces. A *realization* of a polytope P is a polytope $\mathbb{P} \subset \mathbb{R}^d$. A polytope P is *rational* if there is a realization over $\mathbb{Q}$, and irrational if not.

Question: do there exist irrational polytopes? Yes!

Theorem 1.2 (Steinitz)**.** *All* 3*-dimensional polytopes are rational.*

Theorem 1.3 (Perles, Mrev, Richter-Gebert)**.** *There exist irrational polytopes.*

Additionally, all simple and simplicial polytopes are rational: for in a simplicial polytope, we can perturb the vertices slightly to be rational with no problems (NOTE: highly sensitive to the fact that each face is a triangle/simplex). For simple polytopes, we perturb their hyperplanes to be rational. Then their intersections are rational as well. As was pointed out, you can't perturb vertices on a facet without changing it, unless the facet in question is a simplex.

2 Point and line configurations

Take a set of lines and points which lie on these lines (not necessarily at the intersections). We denote the points by $X = \{x_1, \ldots, x_n\}$ and the lines by $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$. Then we can encode the combinatorial data as a bipartite graph where (x_i, ℓ_j) if and only if $x_i \in \ell_j$.

Question: given such a graph $G = (X \sqcup \mathcal{L})$, does there exist a point and line configuration having G as its graph of incidences?

In general, this is a hard question, as often points are subtly forced onto lines (for example, Desargue's theorem, which Igor proved by slightly lifting the middle line and looking at intersecting planes). As another example, the Fano plane (Fig 12.3 in Igor's book, p. 112) has a realization in $\mathbb{F}_2$, but not in the reals:

Theorem 2.1 (Gallai-Sylvester)**.** *In a point and line configuration (where we draw all possible lines through 2+ points) over $\mathbb{R}$, some line must contain either every point, or ≤ 2.*

The Fano plane clearly violates this condition.

Lemma 2.2 (Perles)**.** *There exists an irrational configuration of points and lines.*

Proof. Consider the Perles configuration (Fig 12.4, p. 112 of Igor's book), with vertices $1, 2, \ldots, 9$. We argue by contradiction: assume there exists a rational realization. First, we project the line 1234 to a line at infinity. □

1 Universality theorems

Let A be an algebraic closure of $\mathbb{Q}$, and let k be a proper subfield of A.

Theorem 1.1 (Mnëv)**.** *There is a point and line configuration which is realizable over A, but not realizable over k.*

1.1 Ruler and compass constructions

A trip back in time: Gauss was interested in whether a regular n-gon can be constructed with a ruler and compass.

Theorem 1.2 (Gauss)**.** *The regular heptagon cannot be obtained in such a way.*

People were not impressed with negative results at this time however, and so:

Theorem 1.3 (Gauss)**.** *The regular* 17*-gon is constructible.*

In general, a regular n-gon can be constructed if and only if

$$n = 2^r p_1 \cdots p_k, \tag{1}$$

where the p_i are distinct Fermat primes, i.e. $p_i = 2^{2^i} + 1$.

Theorem 1.4 (Euler)**.** $2^{2^5} + 1$ *is not a prime.*

Today, we'll prove theorem 1.3 "in a strange way."

Definition 1.5. Let $k_0 = \mathbb{Q}$, and $k_{i+1} = k_i[\sqrt{a_i}]$ for some $a_i \in k_i$. If $z \in k_r$ for some r, we call z a *geometric number.*

Theorem 1.6. *A number z is geometric if and only if z can be constructed using ruler and compass.*

Together with the fact that $\cos \frac{\pi}{17}$ is a geometric number, Theorem 1.6 immediately implies Theorem 1.3.

It suffices to show that given positive geometric numbers a, b, we can construct $a+b, a\cdot b, a/b, \sqrt{a}$ (see chapter 12 of Igor's book for one set of constructions that does this).

1.2 Proof of universality for points and lines

Igor shows us how to add, and multiply using point and line configurations (see chapter 12 for the constructions). Consider a polynomial $f(x) = a_n x^n + \cdots + a_0$, with a_i integral. If we start with the point x, we can use addition and multiplication to construct the point $f(x)$.

By declaring that $f(x)$ must be equal to the origin point in our configuration, we obtain a configuration that is only realizable if we have access to an x satisfying $f(x) = 0$.

I should note that this is explained in more detail in chapter 12 of Igor's book.

Theorem 1.7 (Mnëv)**.** *For all algebraic varieties X, there is a point and line configuration $\mathcal{L}$ such that $M(\mathcal{L}) = X \times (\mathbb{C}^*)^m$.*

1.3 Irrational polytopes

Theorem 1.8 (Mnëv)**.** *For all X, there is a polytope P such that $M(P) = X \times \cdots$ ($M(P)$ is the space of realizations of P).*

Lawrence construction proof. Let $\mathcal{L} = \{(p_1, \ldots, p_n), (\ell_1, \ldots, \ell_m)\}$. Lift the points $p_i = (a_i, b_i) \in \mathbb{R}^2$ to $(1, a_i, b_i) \in \mathbb{R}^3$. Now, we set P equal to the convex hull of $\{v_i, w_i\}$, where $v_i, w_i \in \mathbb{R}^{3+n}$ and

$$v_i = (p_i, e_i); \quad w_i = (p_i, 2e_i). \tag{2}$$

Suppose that some nonzero combination of v_i and w_i is equal to 0. This gives us a linear relation on our points p_i. Because we have lifted into $\mathbb{R}^3$ (another projective trick), this corresponds to a colinearity condition. □

1 Linkages and applications

Today's lecture draws heavily from chapter 13 of Igor's book, so I'm only going to include things that are not there. He's got all the figures he drew today in that chapter.

Theorem 1.1 (Kapovich, Millson)**.** *"Nearly" any semi-algebraic set can be drawn using a bar and joint linkage.*

Example of a semi-algebraic set: points satisfying $x^2 + y^2 \leq 1$ and $x, y \geq 0$. We're not gonna prove this one.

Igor first builds an adder $x \mapsto x + c$, then a multiplier $x \mapsto cx$ (this last one is a pantograph). Additionally, Igor shows us how to do vector addition $\vec{z} = \vec{x} + \vec{y}$. Multiplying x and y is more difficult. First Igor builds an inverter, then exploits the fact that inverters are sufficient for multiplication, via

$$\frac{1}{z-1} - \frac{1}{z+1} = \frac{2}{z^2-1}. \tag{1}$$

This allows us to square $z \mapsto z^2$, at which point we have products via

$$xy = \frac{1}{2}[(x+y)^2 - x^2 - y^2]. \tag{2}$$

1.1 How this changed our lives

Before computers, people still needed to do calculations. For example, the *planimeter* gadget was invented: a linkage that computes the area inside a simple closed curve. In 1928-31, Vannevar Bush at MIT created the differential analyzer, a mechanical computer for solving differential equations.

1 Steinitz's theorem

Today's lecture brought to you by chapter 11 of Igor's book. Recall

Theorem 1.1 (Balinski)**.** *The graph of a* 3*-dimensional polytope* P *is planar and* 3*-connected.*

Theorem 1.2 (Steinitz)**.** *Every* 3*-connected planar graph* G *is the graph of a convex rational polytope.*

Corollary 1.3. *Every* 3*-connected planar graph has a unique embedding into* S^2*.*

Igor will give four proofs of the Steinitz theorem.

1.1 Proof 1

Section 11.2 of Igor's book. We apply operations to our graph G to reduce it to K_4. Our operations are the $Y\Delta$ transformation and its inverse ΔY, along with the removal of sequential and parallel edges (Figure 11.1).

Main Lemma 1.4 (Steinitz)**.** *Every planar* 3*-connected graph can be reduced to* K_4 *by this set of transformations.*

Igor shows us the reductions on the graph of the cube to K_4 (Figure 11.2).

Lemma 1.5. *The Main Lemma implies the Steinitz theorem.*

Proof of Lemma 1.5. The $Y\Delta$ moves correspond to slicing a corner of a polytope off with a hyperplane, and its inverse of "joining up" three points of a triangular face (may require a projective transformation). Thus the Main Lemma states that we can start with a simplex and build a polytope with any graph desired. □

Lemma 1.6. *If a planar graph* G *goes to* K_4*, then so will every minor of* G*.*

"Fairly obvious."

Lemma 1.7. *If* G *is planar, then it is a minor of some* $k \times n$ *grid graph.*

"Even easier."

Lemma 1.8. *Every* $k \times n$ *grid graph goes to* K_4*.*

Proof of Lemma 1.8. Starting with a $k \times n$ grid, we turn two opposite corners into diagonals. We move one of these diagonals upwards to the top (sublemma in class), at which point we can kill it using an easy ΔY transformation. Then repeat along the bottom row, until we've reduced to a smaller case, and it's some induction.

It's clear that the 3×3 graph can be turned into K_4. □

There's a slight lie in Lemma 1.6: obviously some extreme minors cannot get to K_4. It turns out that you need to add the stipulation that we take a 2-connected minor (not 3, it's too restrictive), but then this is still not enough. Ziegler ended up modifying the proof to be phrased inductively, which eliminates these scares.

1 More on Steinitz

Recall:

Theorem 1.1 (Steinitz)**.** *Every* 3*-connected planar graph* G *is the graph of a convex polytope* $P \subset \mathbb{Q}^3$*.*

1.1 Steinitz theorem for triangulations

We prove this by induction on $n = |V|$. By Euler's formula we have:

$$|E| = 3n - 6. \tag{1}$$

In particular, we observe that there must be a vertex having degree $3, 4$, or 5. For the case of a degree 3 vertex: remove that vertex, realize the polytope by induction, then add a little pyramid when putting the vertex back. For degree 4, we have a quadrilateral formed by the neighbors of v. Remove v, put a diagonal in, realize the polytope, put a pyramid in with apex slightly above the diagonal.

Finally, we consider the case of a degree 5 vertex. Now the neighbors form a pentagon. Remove v, triangulate the pentagon, and realize the polytope. Then place above the orange line but below the purple line in Figure .

"Why do some theorems need many proofs?" In this case, we have a goal: the quantitative Steinitz theorem. Having more methods means having more chances of proving it.

1.2 Quantitative Steinitz theorem

For all G, we'd like to find the minimum bounding box containing P with integer vertices and $G(P) = G$.

Conjecture 1.2. *For triangulations* G*, there is some constant* c *so that a* $n^c \times n^c \times n^c$ *bounding box suffices, where* n *is the number of vertices of* G*.*

By contrast, our $Y\Delta/\Delta Y$ construction from last time requires an enormous bounding box: O.S. proved that our construction is $e^{O(n^3)}$.

The triangulation proof above "gives at best an exponential bound."

Theorem 1.3 (Demaine-Schulz)**.** *If there is a sequence* $G \to K_4$ *of type* 3 *(removing a degree* 3 *vertex each time), then we can build a bounding box with dimensions* $O(n^5) \times O(n^5) \times O(n^5)$*.*

Sketch of idea: these vertex removals form a ternary tree. Allocate space on your simplex for the big branches of the tree.

The problem is the degree 5 case: our final step is very restrictive.

2 Tutte's spring theorem and Maxwell-Cremona construction

For a planar and 3-connected G and a convex polygon Q, we create a *spring embedding* of G by letting the outside face of G be Q with vertices pinned, and if all inside edges are springs, then the interior vertices are in equilibrium.

Theorem 2.1. *For any spring weights, there is a unique spring embedding. Furthermore, this embedding is planar and every interior face is a convex polygon.*

Physics intuition helps a bunch here: Hooke's law for a spring states that

$$F = k \cdot \ell, \tag{2}$$

and when we integrate we obtain the energy:

$$E = \frac{1}{2} k \cdot \ell^2. \tag{3}$$

For today, Igor wants $k = 1$ for every edge.

Lemma 2.2. *For every connected graph G, there is a unique spring embedding equilibrium.*

The intuition is clear: eventually the springs stop wiggling. To make it rigorous, consider the energy function

$$E = \frac{1}{2} \sum_{(v_i, v_j) \in E(G)} (x_i - x_j)^2 + (y_i - y_j)^2. \tag{4}$$

Our equilibrium condition corresponds to solving a system of linear equations. If the solution space is 1-dimensional or higher, we would have no minimum energy.

1 Maxwell-Cremona theory

Last time we discussed the *Tutte embedding* of a planar 3-connected graph G. Let $w_{i,j} : E \to \mathbb{R}^+$.

Theorem 1.1 (Maxwell 1864)**.** *Any projection of a 3-dimensional polyhedral surface is a (generalized) Tutte embedding. That is, there exist weights $w_{i,j}$ such that the projection is a Tutte embedding.*

Theorem 1.2 (Cremona)**.** *For any generalized Tutte embedding $p : V \to \mathbb{R}^2$, there exists a 3-dimensional polyhedral surface which projects onto p.*

1.1 A (slightly dishonest) explanation

"Normals of hyperplanes are dual to the edge vectors of the graph."

Theorem 1.3. *Let P be a convex polytope, and let u_i be the normal to face F_i. Let a_i be the area of F_i. Then $\sum a_i u_i = 0$.*

Proof. □

2 Circle packings

Question: can you draw every planar graph as the dual of a circle packing?

Theorem 2.1 (Koebe-Andreev-Thurston)**.** *Yes.*

Theorem 2.2 (Schramm 1988)**.** *For every 3-connected planar graph G, there is a convex polytope P with $G(P) = G$ and P is "midscribed" around the unit sphere (all edges are tangent to S^2). Moreover, such an embedding is unique up to Möbius transformations.*

Next time: proof using variational principle.

1 Basic variational principle

Most of the content today is from Chapter 9 of Igor's book.

Theorem 1.1 (folklore)**.** *Suppose $P \subset \mathbb{R}^d$ is a convex polytope, and z is a point in P. Then there is a facet F such that the orthogonal projection of z onto the affine span of F lies inside of F.*

"Obvious," why? There's a physical proof: make z your center of mass using some materials, then set your polytope P on the ground. "But how do you know physics works in dimension 112?" Let's postpone answering that for a second. "How do you know it doesn't roll forever?" Answer: the height of z is decreasing with each roll (informally, it's losing potential energy) (see Figure 9.1 if you don't see why). This gives us an idea of how to reinforce the physical intuition.

Variational principle proof. Let $d(F)$ be the orthoganl distance from z to the affine span of F. We argue that the F which minimizes d has the desired facet property. □

Theorem 1.2. *For any polygon $Q \subset \mathbb{R}^2$ with z the center of mass of Q, there are at least 2 faces which z projects orthogonally onto.*

Note that for $z \in Q$ general, it's false: see Figure 9.3 of Igor's book.

Lemma 1.3. *Let $P, Q \subset \mathbb{R}^2$ be convex plane sets with $cm(P) = cm(Q)$ and $area(P) = area(Q)$. Then the boundaries of P, Q intersect in at least 4 points.*

Proof. By contradiction: obviously the number of intersection points must be even. Suppose there are only 2 points. Draw the line segment between these points. Then the intersection $X = P \cap Q$ satisfies

$$cm(P) = \alpha cm(P \setminus X) + (1 - \alpha) cm(X), \tag{1}$$

$$cm(Q) = \alpha cm(Q \setminus X) + (1 - \alpha) cm(X). \tag{2}$$

□

Proof of Theorem **??**. Draw a circle C centered around $z = cm(Q)$ satisfying $area(C) = area(Q)$. Since C intersects Q at least 4 times, □

Theorem 1.4. *In $d = 11$, there is a simplex with only one stable equilibrium. Furthermore, there is a simplex that "rolls" onto every single facet.*

1.1 Billiards

Theorem 1.5. *Let Q be a smooth convex figure. Then there are at least two double-perpendiculars.*

Proof. The first one is easy: just take the diameter. Finding the second one is significantly more tough: we have to use a minimax argument (See the proof of Theorem 9.10). □

1.2 Birkhoff's theorem

These double perpendiculars are special cases of billiard trajectories, specifically those of size 2.

Theorem 1.6 (Birkhoff)**.** *Every convex set $Q \subset \mathbb{R}^2$ has at least two closed billiard trajectories of size p ("p is prime, but it's not really important").*

Proof. Take points $x_1, \ldots, x_p \in \partial Q$. Define $f(x_1, \ldots, x_p)$ to be the sum of the distances between successive points. Let $z_1, \ldots, z_p$ be a set of points that maximize f. Then we claim that $z_1, \ldots, z_p$ is our desired billiard trajectory.

To see this, we must verify the equal angle property of a reflection. Using the fact that x_i maximizes the sum of distances, all points in Q lie in the ellipse with focal points x_{i-1}, x_{i+1}. In particular, the ellipse and Q have the same tangent at x_i. By the reflection property of an ellipse, the angles are equal. $\square$

1 Variational principle

1.1 Monostatic polytope

There is a convex polytope in $\mathbb{R}^3$ which only rests on one face. An easy construction is the "sliced cylinder" in Figure 9.5.

1.2 The Gömböc

The aforementioned cylinder has a stable facet (local min), but three more non-stable equilibria: balance it on the top edge (saddle), or on the sides (local maxes). *Arnold's conjecture* states that there exists a convex body having only one local min and one local max.

Theorem 1.1. *Arnold's conjecture holds: the Gömböc has one local max, one local min, and no saddle equilibria. Furthermore it can be done as a polytope (?).*

Why is it called a Gömböc? It's the Hungarian name for the "clown who won't stay down" toy (Russian: Heblg[uw]ka http://ru.wikipedia.org/wiki/

2 Closed geodesics

For a surface A, we can think about curves γ which provide the shortest length from x to y on ∂A. A *geodesic* γ is a curve which is "locally the shortest." In particular, for every $z \in \gamma$ there are $x, y \in \gamma$ such that z lies between x and y, and the restricted γ is the shortest curve from x to y.

Problem: find closed geodesics on a two dimensional surface S. *Simple* closed geodesics (no self-intersection) are of interest.

Theorem 2.1 (Lusternik-Shnirelman)**.** *For every smooth convex body $A \subset \mathbb{R}^3$, the boundary $S = \partial A$ has at least* 3 *simple closed geodesics.*

Theorem 2.2 (Morse)**.** *An ellipsoid E with distinct axis lengths/radii has exactly* 3 *simple closed geodesics.*

The proof of Theorem 2.1 is hard, and we won't do it. Let's get one geodesic though: think of a homotopy of circles starting at some base point and ending at some other point. Specifically, let $C(t)$ be a family of closed curves, with $t \in [0, 1]$. Let

$$C = \min_{\{C(t)\}} \max_{t\in[0,1]} |C(t)|. \tag{1}$$

Here we are minimizing over all possible families of curves. What this really is, is a rubber band that won't slide when you let go. Thus this curve C is a closed geodesic.

3 More billiards

Conjecture 3.1. *Every triangle has a closed geodesic.*

This is "high school obvious, by bouncing around the altitudes," but it's totally open for obtuse triangles where you can't get away with that.

3.1 Quasi-billiard trajectories

What happens if we let our billiard path to hit corner x with angle γ at angles α, β but demanding that $|\alpha - \beta| \leq \pi - \gamma$?

Theorem 3.2 (quasi-Birkhoff)**.** *For a polygon P, there exists a closed quasi-billiard trajectory.n*

These trajectories are useful, for example for proving Theorem 2.1 in the polytopal case.

Today's lecture draws from Chapters 10 and 25.

1 Curvature of convex polyhedra

Recall:

Theorem 1.1 (Poincaré,Birkhoff,Lusternik,Shnirelman)**.** *Every smooth convex body has at least three simple closed geodesics.*

The result is tight for an ellipsoid.

Theorem 1.2 (Pogorelov)**.** *Every convex body has at least* 3 *(simple closed) quasi-geodesics.*

A quasi geodesic is a path composed of piecewise geodesic segments, and at each corner we have that the two corner angles α, β both are less than or equal to π (if that sounds crazy, think about the angles at the vertex of a polytope).

A combinatorial proof of Pogorelov's result exists, "but it's too complicated for today."

Claim 1.3. *Random tetrahedra do not have simple closed geodesics.*

"Random" is purposefully left ambiguous.

Proof. There are possibly two closed geodesics that we can have on a tetrahedron: triangles around three faces, or quadrilaterals around all four.

In the first case, take the angle sum of the face angles at the isolated vertex. We express this in terms of the three small triangles formed by our geodesic:

$$\Sigma = 3\pi - \pi - \pi - \pi = 0 former from all triangles, latter from geodesic condition \tag{1}$$

The 3π comes from the angle sums of all three triangles, while each copy of $-\pi$ Since the angle sum can't be zero, this is a contradiction.

For the second case, consider the angle sums from one pair of isolated vertices:

$$\Sigma = 2 \cdot (2\pi) + 2 \cdot \pi - 4 \cdot \pi = 2\pi. \tag{2}$$

This angle sum however is a measure zero condition, so a random tetrahedron won't satisfy this condition. □

Definition 1.4 (Descartes)**.** Suppose C is a cone with vertex O. Define $\omega(C)$ to be equal to $2\pi - \sum(\angle)$, the sum of the angles of the cone. Call this the *curvature* of C at O.

Theorem 1.5 (Descartes)**.** *For every convex polytope* $P \subset \mathbb{R}^3$*,*

$$\sum_{v \in V(P)} \omega(C_v) = 4\pi. \tag{3}$$

Boring proof. Triangulate everything, thus assuming that P is simplicial. If not, we can just triangulate each face and we don't care if faces are parallel.

Then we have

$$\omega(P) = \sum_{v \in V(P)} \omega(C_v) = 2\pi n - \sum(\angle). \tag{4}$$

By Euler's formula, there are $2n-4$ faces, and since each is a triangle the angle sum above is equal to $(2n-4)\pi$. The result is proved. $\square$

Gauss' proof. Assume every cone has three faces, i.e. we have a simple polytope. For every vertex, construct a second cone

$$C_v^* = \{\vec{x} : \langle x, y\rangle \leq 0 \forall y \in C_v\}. \tag{5}$$

Redefine ω_v to be the surface area of of $C_v^* \cap S^2$, where S^2 is a unit sphere.

We claim of course that $\omega_v = \omega(C_v)$. We also claim that $\sum \omega_v = 4\pi$. This latter claim is not hard to see: if we give the dual cones C_v^* a common origin point, they form a disjoint union (modulo boundaries) of $\mathbb{R}^3$. Hence, the sum is equal to the area of S^2. To see the disjoint union property, consider the hyperplane orthogonal to a specified point, to determine which cone the point lies in.

Lemma 1.6 (Girard's formula)**.** *The area of a spherical triangle is equal to $\alpha + \beta + \gamma - \pi$, where α, β, γ are the angles of the triangle.*

Proof of lemma. See Chapter 41 of Igor's book for an inclusion-exclusion style proof. $\square$

Now to see the first claim, we note that the condition defining C_v^* turns the face angles into dihedral angles, i.e. $\alpha \to \pi - \alpha$. Then we immediately have

$$\omega(C_v) = 2\pi - \alpha - \beta - \gamma = (\pi - \alpha) + (\pi - \beta) + (\pi - \gamma) - \pi = \omega_v. \tag{6}$$

$\square$

Today's lecture brought to you by Chapter 35 of Igor's book.

1 Alexandrov's curvature theorem and the Weyl problem

Suppose we have $P \subset \mathbb{R}^3$ with vertices $v_1, \ldots, v_n$. Let R_i be the rays from the origin to v_i.

Question: given the rays R_i, what can be said about the curvatures $w_i = \omega(v_i)$?

Theorem 1.1 (Alexandrov, 1930s)**.** *Given rays $R_1, \ldots, R_n$, a vector $w = (w_1, \ldots, w_n)$ is a vector of curvatures if and only if:*

1. *$w_i > 0$ for all i,*
2. *$\sum w_i = 4\pi$ (Gauss-Bonnet),*
3. *For every subset $I \subset [n]$, we have*

$$\sum_{j \notin I} w_j > \omega(C_I), \tag{1}$$

where C_I is the cone of the rays R_i with $i \in I$.

Moreover, such $P = conv(v_1, \ldots, v_n)$ is unique up to expansion.

In other words, there is a bijection between ray-polytopes modulo expansion, and "valid" curvature vectors. This last comment on uniqueness turns out to be an important factor in the proof.

Lemma 1.2. *Suppose P Q are ray-polytopes that lie on the same set of rays, with vertices v_i and w_i respectively. Then if $\omega(v_i) = \omega(w_i)$ for all i, we have that P is an expansion of Q.*

Proof of lemma. Without loss of generality, assume Q is small enough to fit completely inside P. We grow Q until a vertex of Q first touches the boundary of P (necessarily at a vertex of P). Now if P, Q are not equal, there is a contact vertex $w_i = v_i$ where the cone of w_i lies strictly inside the cone of v_i. But curvature is a (strictly) monotonic function, which is easily seen by using the orthogonal cones of last lecture. Thus we would have $\omega(w_i) > \omega(v_i)$, a contradiction. $\square$

Lemma 1.3. *Let A, B be d-dimensional manifolds, and let $\varphi : A \to B$ satisfy the conditions*

1. *Every connected component of B intersects $\varphi(A)$.*
2. *φ is injective.*
3. *φ is continuous.*

4. φ is proper (preimages of compact sets are compact).

Then φ is a homeomorphism.

Proof of Alexandrov. Let A be the space of ray-polytopes P up to expansion, and let B be the space of valid curvature vectors. Of course, we let $\varphi(v_1, \ldots, v_n) = (w_1, \ldots, w_n)$. We then show that φ satisfies the conditions of the above lemma, which will prove our theorem.

It is better to consider φ as a function of the distances r_i along rays R_i (remember that our rays are fixed from the start). Conditions 1 and 2 of the lemma are trivial. Condition 3 is a consquence of Lemma 1.2. Thus, it remains to check that φ is proper. To do this, we will need to use the third condition on validity of a curvature vector.

To ignore expansion, designate that $\prod r_i = 1$. To prove properness, we will show that a convergent sequence of curvature vectors $\varphi(P_j)$ has a limit point $P \in A$, by contradiction. $\square$

1.1 Relation to the Weyl problem

Given now a *smooth* manifold, how much does the Gaussian curvature determine a convex body? Alexandrov's student Pogorelov extended our above theorem to *all* convex bodies. Another person used analytic methods to prove the smooth version around the same time.

Chapter 15 of Igor's book would be good reading for this lecture.

1 Hilbert's third problem

Hilbert's famous problem list was delivered in 1900. The third problem was the first one solved.

Definition 1.1. Suppose $P, Q \subset \mathbb{R}^d$ are convex polytopes of equal volume. We say that P, Q are *scissors congruent* and write $P \sim Q$ if $P = \sqcup P_i, Q = \sqcup Q_i$ where the unions are finite and $P_i \simeq Q_i$ (congruent).

1.1 Problem statement

True or false: are all polytopes in $\mathbb{R}^3$ of equal volume scissors congruent?

Theorem 1.2 (Dehn 1902)**.** *The unit cube is not scissors congruent to the regular tetrahedron of volume* 1*. That is,* $C \not\sim \Delta$*.*

1.2 Why three dimensions?

Why start with $\mathbb{R}^3$?

Theorem 1.3 (Bolyai, Gerwein)**.** *All equal-area convex polygons in* $\mathbb{R}^2$ *are scissors congruent.*

Proof. Start by cutting P, Q into triangles $P = \sqcup P_i, Q = \sqcup Q_j$. Denote by α_i the area of P_i, and let β_j be the area of Q_j. Now refine our cuts into triangles P_{ij}, Q_{ji} having area $\alpha_i\beta_j$. We have now reduced our problem to showing that any two triangles of equal area are scissors congruent.

Proving that two triangles are scissors congruent is easy: First, chop off the top of a triangle and move it onto the side, in order to convert it to a parallelogram. Note that two parallelograms with one side length in common are scissors congruent (see Figure 15.2). (Then the proof finishes... how?)

(In fact, we can get more general: there is a lemma that states that if G acts on X such that P, Q are fundamental regions, then $P \sim Q$.) $\square$

1.3 Area and volume

This above proof tells us an easy way to define the area of any polytope, by using scissors congruence to relate it to a rectangle. To compute the volume of a regular tetrahedron, people typically use calculus-like results.

1.4 Proof

Proof. Let P have dihedral angles $\alpha_1, \ldots, \alpha_N$. Call P *fortunate* if there are rationals $c_1, \ldots, c_N$ such that

$$c_1\alpha_1 + \cdots + c_N\alpha_N = \pi. \tag{1}$$

Lemma 1.4 (Main Lemma)**.** *If $P \sim C$, then P is fortunate.*

Before we prove the Main Lemma, let us observe that it implies the theorem: all dihedral angles of the regular tetrahedron are equal to $\alpha = \arccos(1/3)$. However, α/π is not a member of $\mathbb{Q}$. $\square$

1.4.1 First "proof"

For the Main Lemma, Igor presents a "correct but also incorrect" proof as follows:

Proof 1 of Main Lemma. Let $P = \sqcup P_i$, $C = \sqcup C_i$ be a decomposition of P, C into congruent polytopes. Since $P_i \simeq C_i$ the sum of all dihedral angles is equal:

$$\sum_{all P_i} \angle = \sum_{all C_i} \angle. \tag{2}$$

Interior dihedral angles contribute multiples of 2π to this sum. Meanwhile, the angles corresponding to the surface of the cube contribute a multiple of $\pi/2$. Thus, we can take the surface angles of P and create a fortunate combination. $\square$

What's wrong with this? Igor brings up another theorem to illustrate:

Theorem 1.5 (Bricard 1897)**.** *The tetrahedron and cube are not scissors congruent,* even if you consider dissections*.*

Our above proof only works if our cuts are always face-to-face, as we may end up counting the same dihedral twice in a general dissection.

Proof 2. To fix this, let e be an edge of the dissection, and consider the modified sum

$$\sum_{e \in P_i} |e| \angle e = \sum_{e \in C_i} |e| \angle e. \tag{3}$$

We get the same result, *but only if $|e|$ is always rational.* $\square$

So we can do it for face-to-face triangulations, and we can do it for rational dissections. Our final proof closes the gap:

Proof 3. Let's try to find a rational valued function $f(|e|)$ to help us. What will this function need to do? We'd need to satisfy

$$\sum_{e \in E} f(e) = \sum_{e' \in E'} f(e'); \sum f(e) \angle e = c\pi. \tag{4}$$

Now, this first sum equality is a system of equations in the unknowns $f(e), f(e')$. We know at least one solution exists via $f(e) = |e|$ However, once one solution exists we know that a rational function exists (?). $\square$

1.5 Final comments

Hidden behind the scenes here is a tensor product $\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z}$, along with the function $\phi(P) = \sum e \otimes \angle(e)$. This more algebraic approach can be used to show more things, such as the "inverse problem."

1 Scissors congruence: combinatorial approach

Last time was a double counting argument, but we can do even better. Idea: use valuations! Define a map φ from all d-dimensional polytops P to $\mathbb{R}$, satisfying:

* φ is invariant under scissors congruence. Namely, if $P \simeq P'$ then $\varphi(P) = \varphi(P')$. Additionally, we'd like that if $P = P_1 \sqcup P_2$ then $\varphi(P) = \varphi(P_1) + \varphi(P_2)$.

1. $\varphi(C) = 0$, where C is the cube, and
2. $\varphi(\Delta) \neq 0$, where Δ is the tetrahedron.

Technically, this first one is the only thing we need to call our function a (symmetric) valuation.

Example: $\varphi(P) = c \cdot vol(P)$ is a valuation for any c.

Definition 1.1. An *additive function* $f : \mathbb{R} \to \mathbb{R}$ is any function which satisfies

$$f(a+b) = f(a) + f(b).$$

Today, we require of an additive function that

$$f(\pi) = 0. \tag{π}$$

Definition 1.2. For an additive function f, the *Dehn valuation* is defined by

$$\varphi(\Delta) = \sum_{e \in \Delta} |e| \cdot f(\angle_e).$$

Theorem 1.3. *The Dehn valuation is a symmetric valuation.*

Before proving this, let's see that the theorem implies Hilbert's third problem. Let $\theta = \arccos(1/3)$ and let us demand that $f(\theta) = 1$.

Lemma 1.4. *For* any *polytope P, we have*

$$\varphi(P) = \sum_{e \in P} |e| f(\angle_e).$$

Thus for the cube, we have

$$\varphi(C) = 12 f\left(\frac{\pi}{2}\right) = 0,$$

and for the tetrahedron we have

$$\varphi(\Delta) = 6|e|f(\theta) = 6|e| \neq 0.$$

Therefore, it would be impossible for the cube and the tetrahedron to be scissors congruent.

1.1 Theorem implies lemma

Assuming our theorem, take any polytope P and triangulate it into simplices Δ_i. Then we have that $\varphi(P) = \sum \varphi(\Delta_i)$. More specifically, we have

$$\sum_{\Delta_i} \sum_{e \in \Delta_i} |e| f(\angle_e).$$

But all the interior dihedrals add up to rational multiples of π, so only the dihedrals of P remain. This proves that the lemma follows from our theorem.

To prove our theorem we will use the following fact:

Theorem 1.5 (Ludwig, Reitzner). *Let T_1, T_2 be triangulations (dissections into simplices) of $P \subset \mathbb{R}^d$. Then $T_1 \to T_2$ via a sequence of 2-moves (see Figure 17.3 of Igor's book).*

Igor emphasizes that it can be non-obvious to move between two triangulations (see Figures 17.2 and 17.4).

Lemma 1.6. *In $d = 3$, φ is invariant under 2-moves.*

Proof of lemma. Consider Figure 17.3. The dihedrals coming from the red edges contribute nothing due to them adding up to a rational multpile of π. Meanwhile the original edges of the tetrahedron add up properly, due to things being multiplied by edge lengths. □

The proof of Ludwig-Reitzner is similar to many topology style proofs.

$\mathbb{R} \to \infty$

1 2-move connectivity of triangulations

Definition 1.1. A 2-*move* is a transfer as in Figure 1, or its higher dimensional analogues.

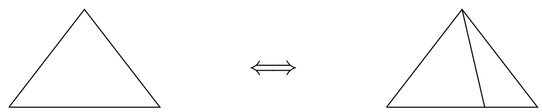

Figure 1: A typical 2-move

Theorem 1.2 (LR 2006)**.** *Let P be a convex polytope in $\mathbb{R}^d$ with triangulations (not necessarily face-to-face) T_1, T_2. Then T_1, T_2 are connected by a series of* 2*-moves.*

Lemma 1.3. *"Flips" are obtained as* 2*-moves.*

For the proof, see Figure 17.4 in Igor's book.

Lemma 1.4. *Every two full triangulations of a convex polygon P are connected via flips.*

Definition 1.5. A *star triangulation* is a full triangulation of a polygon P where all triangles have a common vertex.

Our proof of Lemma 1.4 is just the observation that we can (reversibly) reduce any two triangulations to the same star triangulation.

Lemma 1.6. *Let $P = P_1 \sqcup P_2$, and let T_i be star triangulations of P_i. Then $T_1 \sqcup T_2$ is* 2*-move connected to a star triangulation of P.*

Proof of Lemma 1.6. A simple reduction. See the proof of Lemma 17.11 and Figure 17.5 in Igor's book. □

Our main theorem will be proved by a tricky induction.

Proof of Theorem. We want to show that every triangulation T is equivalent to a star triangulation of P. We induct on the number of polygons in polygonal dissections Q of P.

Claim 1.7. *Every star subtriangulation of Q is connected to a star triangulation of P. By a subtriangulation we mean that every polygon in the dissection Q is star triangulated.*

Observe that the base case is when Q is just P, and all star triangulations of P are connected. For bigger cases, select an interior edge e and chop P, Q into two pieces $P_1, Q_1; P_2, Q_2$ along the line containing e (see Figure 17.6 in Igor's book).

Observe that the number of polygons goes down from Q to Q_i. To finish, we need to apply Lemma 1.6 to the polygons of Q that were cut, as well as to P_1, P_2. □

Theorem 1.8. *Same result for non-convex polytopes.*

Proof. For a non-convex P with triangulations T_1, T_2, refine T_1 using T_2. For each polytope of this refinement, use our theorem to reduce it to T_i. $\square$

For higher dimensions, we need to argue that our Lemmas still hold. In fact, this is false: Lemma 1.4 is open in $\mathbb{R}^3$, and false in dimensions 5 (or maybe 6).

1 Polytope algebra

Theorem 1.1 (Sydler c.1960)**.** *Suppose $P, Q \subset \mathbb{R}^3$ are convex polytopes such that $volP = volQ$. Further suppose that for all additive functions f, $\varphi_f(P) = \varphi_f(Q)$ (where φ is the Dehn valuation). Then $P \sim Q$ and vice versa (scissors congruence).*

No proof provided: Sydler's proof is really long, and the modern proof while shorter requires k-theory.

Proposition 1.2. *Let Δ be the regular tetrahedron. Then $\Delta \not\sim c_1\Delta_1 \oplus c_2\Delta_2$, for any positive constants c_1, c_2. In particular, Δ is not scissors congruent to the triangular bipyramid.*

1.1 Complementarity lemma

Lemma 1.3. *Suppose $A \oplus B \sim C \oplus D$ and that $A \sim C$. Then $B \sim D$.*

1.2 Tiling lemma

Lemma 1.4. *Suppose P is scissors congruent to $c_1P \oplus c_2P \oplus \cdots \oplus c_mP$ with each c_i positive. Then $P \in \mathcal{R}$, where*

$$\mathcal{R} = \{R : R \sim \alpha \cdot C\}, \tag{1}$$

with C the unit cube. Igor calls such polytopes "Rectifiable."

1.3 One more lemma

Lemma 1.5. *Let $P \subset \mathbb{R}^3$, $\alpha_1, \ldots, \alpha_k > 0$ and $\alpha_1 + \cdots + \alpha_k = 1$. Then*

$$P \sim \alpha_1 P \oplus \cdots \oplus \alpha_k P \oplus R, \tag{2}$$

where R is rectifiable.

Proof. Write $P = \bigoplus \Delta_i$. Since rectifiable polytopes are closed under addition, it suffices to show that the result holds for any tetrahedron Δ_i. This is accomplished via proof by picture: see Figure 16.1 of Igor's book. □

Note that Igor's picture involves a mirror reflection, so we do need to confirm that polytopes are scissors congruent to their mirror images. Note again that it suffices to show this is true on tetrahedra. For a given tetrahedron, inscribe a sphere inside it, and take the barycentric subdivision using the contact points. By the construction of the sphere, these resulting subtetrahedra are symmetric, and we're done (see exercise 15.3).

Theorem 1.6. *Suppose $n \times P = P \oplus \cdots \oplus P$ is rectifiable. Then P is rectifiable.*

Proof. We have

$$n \times P \sim R_1, \tag{3}$$

$$n \cdot P \sim (n \times P) \oplus R_2. \tag{4}$$

Thus we have

$$n \cdot P \sim R_1 \oplus R_2, \tag{5}$$

and so $n \cdot P$ is rectifiable. Since $\mathcal{R}$ is closed under homotheties, we are done. $\square$

1 More on polytope algebra

Recall from yesterday: the question is to determine when polytopes P, Q are scissors congruent in $\mathbb{R}^3$.

Theorem 1.1 (Dehn)**.** *Δ is not rectifiable, where Δ is the regular tetrahedron.*

1.1 Things we already proved

Lemma 1.2. *Let $\alpha_1 + \cdots + \alpha_k = 1$. Then*

$$P \sim \alpha_1 P \oplus \cdots \oplus \alpha_k P + R, \tag{1}$$

for some rectifiable R.

Theorem 1.3. *Suppose $n \times P = P \oplus \cdots \oplus P$ is rectifiable. Then P is rectifiable as well.*

1.2 Today

Theorem 1.4. *If $A \oplus B \sim C \oplus D$ and $B \sim D$, then $A \sim C$.*

Proof. First, note that necessarily $volB = volD$. Let $A' = \frac{1}{n}A$, and similarly for B', C', D'. By Lemma 1.2,

$$A \sim (n \times A') \oplus R_1, \tag{2}$$

$$C \sim (n \times C') \oplus R_2. \tag{3}$$

Now,

$$volR_1 = volR_2 = \left(1 - \frac{1}{n^2}\right) volA. \tag{4}$$

Additionally we have

$$\frac{vol(n \times B')}{volA} \to 0; \quad n \to \infty. \tag{5}$$

The point here is, we can pick n large enough so that

$$R_1 \sim (n \times B') \oplus S \tag{6}$$

(in a nutshell, n/n^3 shrinks so eventually we can fit some copies of B in R_1).

Here comes the hurricane: we have

$$\begin{aligned} A &\sim (n \times A') \oplus R_1 & (7)\\ &\sim (n \times A') \oplus (n \times B') \oplus S & (8)\\ &\sim n \times (A' \oplus B') \oplus S & (9)\\ &\sim n \times (C' \oplus D') \oplus S & (10)\\ &\sim (n \times C') \oplus (n \times D') \oplus S & (11)\\ &\sim (n \times C') \oplus (n \times B') \oplus S & (12)\\ &\sim (n \times C') \oplus R_1 & (13)\\ &\sim C. & (14) \end{aligned}$$

This completes the proof. $\square$

1.3 Some fun theorems

After all this, we get a reward.

Theorem 1.5 (Sydler). *1. P is rectifiable if and only if*

$$P \sim c_1 P \oplus \cdots \oplus c_k P \tag{15}$$

with $c_i > 0, k \geq 2$.

2. P is rectifiable if and only if $P \sim cP \oplus R$ for some rectifiable R.

Corollary 1.6. *Δ is not scissors congruent to multiple smaller copies of itself.*

Corollary 1.7. *Δ is not scissors congruent to a smaller regular tetrahedron and a cube.*

Proof of Theorem 1.5. One direction of each of Sydler's criteria is easy.

For the second statement, by Lemma 1.2 we have

$$cP \oplus R \sim P \sim cP \oplus (1-c)P \oplus R'. \tag{16}$$

Cancelling cP from both sides, we have

$$(1-c)P \oplus R' \sim R. \tag{17}$$

Thus, P is rectifiable.

For the first statement, let $c = c_1 + \cdots + c_k > 1$ and assume the right hand side: by Lemma 1.2

$$cP \sim c_1 P \oplus \cdots \oplus c_k P \oplus R \sim P \oplus R. \tag{18}$$

Ignoring the middle, we have

$$cP \sim P \oplus R, \tag{19}$$

and now we finish by applying the second criterion to conclude that P is rectifiable. $\square$

Theorem 1.8. *Let Δ_1 be the tetrahedron given by the equations*

$$x, y, z \geq 0; x + y + z \leq 1. \tag{20}$$

Let Δ_2 be the "Hill tetrahedron" given by the four vertices

$$\Delta_2 = conv[(0,0,0),(1,0,0),(1,1,0),(1,1,1)] \tag{21}$$

(see Figure 16.2). Then no two of $\Delta, \Delta_1, \Delta_2$ (properly scaled for volume) are scissors congruent.

Proof. Recall that 6 copies of Δ_2 form a cube. Thus, Δ cannot be scissors congruent to Δ_2.

Next, observe that Δ and four copies of Δ_1 can be assembeled into a cube. If $\Delta \sim c\Delta_1$, we would immediately have by Sydler that Δ is rectifiable. Thus, Δ is not scissors congruent to Δ_1.

Also, observe that if we chop off four corner tetrahedra from Δ (3D triforce it), we get an octahedron. The octahedron can be cut up into 8 copies of Δ_1. Thus, we have

$$\Delta = 4 \times \left(\frac{1}{2}\Delta\right) \oplus 8 \times (c\Delta_1). \tag{22}$$

This provides another proof that Δ is not scissors congruent to Δ_1.

Finally, if $\Delta_1 \sim \Delta_2$, then Δ_1 is rectifiable. However, as in the second paragraph the cube minus four copies of Δ_1 gives us a copy of Δ. This would imply that Δ is rectifiable, so we conclude that Δ_1 is not scissors congruent to Δ_2. □

Corollary 1.9. *None of these tetrahedra are scissors congruent to the octahedron Q.*

Proof. First, observe that that Q is not rectifiable, as $Q = 8 \times \Delta_1$. Thus, Δ_2 is not scissors congruent to Q, as Δ_2 is rectifiable. □

2 Second hour

Theorem 2.1. *In Wonderland, Alice is scissors congruent to The Rabbit.*

Definition 2.2. In Wonderland we say $P \simeq cP$ for all positive c. We write $P \asymp Q$ if $P \sim Q$ under this new equivalence

Essentially, we are allowed to grow and shrink polytopes at will in our cutting and gluing. Thus our theorem says that every two polytopes are the same under this equivalence.

Proof. It suffices to show that all tetrahedra are rectifiable. Observe that we have

$$\Delta \asymp 2 \times \Delta \oplus R_1, \tag{23}$$

$$\Delta \asymp 3 \times \Delta \oplus R_2. \tag{24}$$

Cancelling we have

$$\Delta \oplus R_2 \asymp R_1, \tag{25}$$

and by Sydler this implies that Δ is rectifiable. □

This says that size is important.

2.1 Higher dimensions

Claim 2.3. *Q^4 is rectifiable.*

This is a consequence of the following fact.

Proposition 2.4. *Q^4 tiles $\mathbb{R}^4$ periodically.*

Hint for proof: consider D_4. Then we just quote the following theorem.

Theorem 2.5. *If P tiles $\mathbb{R}^d$ periodically, then P is rectifiable.*

Proof. Just re-slice $\mathbb{R}^d$ into parallelepipeds. □

This is an interesting exception, as 4, 2 are the only dimensions in which the cross polytope is rectifiable.

Theorem 2.6 (Jessen)**.** *For every $P \in \mathbb{R}^4$, there is a $P' \in \mathbb{R}^3$ such that $P \sim P' \times [0, 1]$.*

2.2 Monge equivalence

Definition 2.7. For polytopes P, Q in $\mathbb{R}^3$, we say $P \bowtie Q$ if there is a map $\varphi : P \to Q$ such that

1. φ is continuous,
2. φ is volume preserving,
3. φ is piecewise linear.

Such a map φ is called a *Monge map.*

We've already kind of shown that without the continuity criterion, everything would be equivalent (prove all equal volume tetrahedra are the same, then decompose P, Q into equal volume tetrahedra).

Igor poses the following question for fun: take two unit squares with a small square removed somewhere from each. Are they Monge equivalent (see Figure 18.5)?

Theorem 2.8. *If P is PL-homeomorphic to Q and $volP = volQ$, then $P \bowtie Q$.*

In fact, without much more work we get the full theorem.

Theorem 2.9. *If P, Q have the same volume, then $P \bowtie Q$.*

As the first of the above two theorems suggests, it turns out we should focus on the first two criteria.

Lemma 2.10. *There exists a φ satisfying the first two criteria.*

Proof. Consider first for polygons: we can reduce the number of vertices until we reach a triangle (see Figure 18.1).

For higher dimensions, cut our polytopes up into mutual simplicial cones (see Figure 18.2). □

We now know that we can build a PL-homeomorphism φ. We'd like to modify our idea slightly to get a volume preserving φ.

1 Borsuk problem

(Chapter 3 of Igor's book)

Let $X \subset \mathbb{R}^d$ be convex (polytope) with $diam(X) = 1$. Can we partition X into sets $X_1, \dots, X_{d+1}$ such that $diam(X_i) < diam(X)$?

Proposition 1.1. *For $d = 2$ we can partition X into four sets of smaller diameter.*

Proof is by looking at the quadrants of a circle. Unfortunately, we cannot get away with splitting a circle into three pieces: consider an equilateral triangle with side lengths 1 for X.

Theorem 1.2 (Last time)**.** *Let $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^2$ such that every 3 points can be covered with a unit circle. Then all of X can be covered with a unit circle.*

Lemma 1.3. *Suppose that three points x_1, x_2, x_3 have pairwise distances less than 1. Then all three points can be covered with a circle of radius $r = \sqrt{\frac{1}{3}}$.*

Theorem 1.4 (Borsuk conjecture for $d = 2$)**.** *Scott-style proof: figure 3.2 and the surrounding argument.*

Conjecture 1.5 (Borsuk)**.** *We can do this in every dimension.*

This was disproved, "in a terrible horrible no-good way." However, the result holds in three dimensions (remains open in $d = 4, \dots, 200$ or so).

Theorem 1.6. *The Borsuk conjecture holds for $d = 3$.*

Theorem 1.7 (Kahn,Kalai)**.** *For $N = c^{\sqrt{d}}$, there exists an $X \subset \mathbb{R}^d$ such that for all partitions $X = \bigcup_{i=1}^{N} X_i$, one of the pieces X_i has $diam(X_i) = diam(X)$.*

Open problem: improve the lower bound of N to c^d.

2 Why did people believe it?

Apparently lots of people believed that the Borsuk conjecture would hold. The next theorems possibly explain why.

Theorem 2.1. *The Borsuk conjecture holds for any centrally symmetric convex polytope.*

The proof is simple: draw any hyperplane through the center (passing through no vertices).

Theorem 2.2. *The Borsuk conjecture holds for smooth convex bodies.*