206A Fall 2016 Combinatorics on Integer Sequences
Lectures by Professor Igor Pak

Updated: Monday 19th December, 2016 at 18:04 by Bon-Soon

These are rough notes taken and may be riddled with errors. But it provides a sketch of rough ideas presented.

WEEK 0 FRIDAY

1 Question: "What is a formula?"

Consider the Fibonacci numbers $f_1, f_2, \ldots$, what describes it? We have the following options:

(1) By a recursive definition.
(2) By its generating function.
(3) By an explicit formula.
(4) By a summation formula.
(5) By a solution to a particular dynamical system (by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$).

and perhaps one more interesting way,

(6) By counting certain walks on a graph.

2 Walks on graphs.

Often a combinatorial sequence arise from the counting the number of a certain walk on a graph.

For example, consider the graph (with loop) $\Gamma = (V, E)$ with $V = \{0, 1\}$ and $E = \{a = (0,0), b = (0,1)\}$. Let us denote f_n to be the number of length n walks from 0 to 0. We note that

$$\begin{aligned} f_1 &= 1 \quad \text{with walk } 00 \\ f_2 &= 2 \quad \text{with walks } 000, 010 \\ f_3 &= 3 \quad \text{with walks } 0000, 0010, 0100 \\ f_4 &= 5 \quad \text{with walks } 00000, 00010, 00100, 01000, 01010 \end{aligned}$$

(which we see this gives the Fibonacci numbers!)

3 Walks on finite graphs and rational generating functions.

For a sequence (a_n) that arise from counting certain walks on a finite graph Γ, we can say something about its ordinary generating function $A(t) = \sum a_n t^n$.

Theorem 1. *Let Γ be a finite graph, and S, T two vertices in Γ (need not distinct). Denote a_n to be the number of length n walks from $S \to T$. Then its generating function $A(t) = \sum a_n t^n$ is a rational.*

Here rational means in the rational function sense.

Note above theorem does not hold if the graph is note finite. For example, consider the graph $\Gamma = \mathbb{N}$ with the usual edge set $E = \{(i, i+1) : i \geqslant 0\}$. Note that for C_n denoting the number of $0 \to 0$ walks on Γ of length $2n$ is precisely the n-th Catalan number, namely $C_n = \frac{1}{n+1}\binom{2n}{n}$. We remark that $\sum_{n\geqslant 0} C_n t^n = \frac{1+\sqrt{1-4t}}{2t}$ is not rational.

As another example, consider the graph $\mathbb{Z}$ with usual edge set $\{(i, i+1)\}$, and b_n =# of $0 \to 0$ walks of length $2n$. We note that $b_n = \binom{2n}{n}$, and that $B(t) = \sum_{n\geqslant 0} b_n t^n = \frac{1}{\sqrt{1-4t}}$ is also note rational.

Exercise 2. Verify $A(t)$ and $B(t)$ as given above. Can we find a functional relation between $A(t)$ and $B(t)$?

Also, using above generating function, one can show that walks on $\mathbb{Z}$ will visit 0 infinitely often; same with walks on $\mathbb{Z}^2$, but not $\mathbb{Z}^3$. (Phrasing and proof of this?)

4 Rational and algebraic generating functions.

We say a generating function $\eta \in K[[x]]$ is rational if there exists polynomials $p, q \in K[x]$ such that $p + q\eta = 0$, or $\eta = -p/q$.

We can generalize this: A generating function $\eta \in K[[x]]$ is algebraic (an algebraic series) if there exists polynomials $p_0, p_1, \ldots, p_n \in K[x]$ such that $p_0 + p_1\eta + p_2\eta^2 + \cdots + p_n\eta^n = 0$. The least number n such that this equation hold for η is called the degree of η.

For example: Let $\eta = \sum_{n\geqslant 0}\binom{2n}{n}x^n$. One can show that $\eta^2 = 1 + 4x\eta^2$, hence $\eta = 1/\sqrt{1-4x}$. Note: This is equivalent to showing $4^n = \sum_k \binom{2k}{k}\binom{2(n-k)}{n-k}$.

5 What is "combinatorics?"

Is every problem and every sequence combinatorial? Is the following question combinatorial?

Conjecture 3. *There exists infinitely many prime Fibonacci numbers.*

Note for the Fibonacci numbers F_k, we have for integers $m, n \geqslant 3$, $F_m | F_{mn}$.

(Next lecture is next Friday.)

WEEK 1 FRIDAY

6 Classes of generating functions.

We will look at the following classes of generating functions, with their inclusions as follows:

$\mathbb{N}$-rational $\subset$ Rational $\subset$ Algebraic $\subset$ D-finite $\subset$ ADE.

7 Back to walks on graphs: An instance of an $\mathbb{N}$-rational function.

Consider the following theorem:

Theorem 4. *Let Γ be a finite graph, directed, and let S, T be vertices in Γ. Let $a_n = a_n(\Gamma; S, T) = \# \, S \to T$ paths of length n in Γ. Then $A(t) = \sum_{n \geqslant 0} a_n t^n$ is rational.*

Proof idea. Denote $A_I(t) = \sum (\# \, S \to I \text{ path of length } n) \, t^n$. Note that $A_I(t) = t \sum_{J:(J,I) \text{ an edge}} A_J(t)$. This yields a system of linear equations. Solving yields a rational function for A.

Now in fact, we have something stronger:

Theorem 5. *$A(t)$ above is $\mathbb{N}$-rational.*

8 "Schutzenberger principle."

"If $a_n \in \mathbb{N}$ with $A(t)$ rational, and a_n counts 'combinatorial objects', then $A(t)$ is $\mathbb{N}$-rational."

9 Definition of the class of $\mathbb{N}$-rational functions.

Let $\mathcal{A}$ be the class of all rational generating functions such that:

(1) $0, 1, t \in \mathcal{A}$,
(2) $A, B \in \mathcal{A} \implies AB, A + B \in \mathcal{A}$, and
(3) $A \in \mathcal{A}$ and $[1]A = 0 \implies \frac{1}{1-A} \in \mathcal{A}$.

This class $\mathcal{A}$ generated by $0, 1, t$ as above are the $\mathbb{N}$-rational functions.

In fact we have the following characterization of $\mathcal{A}$:

Theorem 6. *Let $\mathcal{F}$ = class of all $A(t) = \sum a_n(\Gamma; S, T) t^n$, then $\mathcal{F} = \mathcal{A}$.*

Example 7. Fibonacci numbers are $\mathbb{N}$-rational. Indeed: For $\Gamma = (V, E)$ with $V = \{0, 1\}$ and $E = \{(0,0), (0,1), (1,0)\}$, then then number of $0 \to 0$ paths on Γ gives Fibonacci numbers.

10 Algebraic generating functions.

Recall we can say $A(t)$ algebraic if there exists $c_i \in \mathbb{Z}[t]$ such that $c_0 A^r + c_1 A^{r-1} + \cdots + c_r = 0$.

Example 8. Consider $a_n = \frac{1}{n+1}\binom{2n}{n}$, with generating function $A(t) = \sum_{n \geqslant 0} a_n t^n$. Here a_n counts the number of binary trees with n vertices. For instance $a_3 = 5$. By taking a nontrivial binary tree T, one has a left subtree T_l and a right subtree T_r joined to the root of T. Here both T_l and T_r are both binary trees themselves. Hence we can get $A = 1 + tA^2$. This shows $A(t)$ is algebraic.

Note $A(t) = \frac{1 - \sqrt{1-4t}}{2t}$ with coefficients $a_n \sim C \frac{4^n}{n\sqrt{n}}$.

Interestingly we also know the parity of a_n very well:

Theorem 9. *(Kummer) For $a_n = \frac{1}{n+1}\binom{2n}{n}$, we have*

$$a_n \mod 2 = \begin{cases} 1 & \textit{if } n = 2^k - 1 \\ 0 & \textit{else} \end{cases}$$

.

Proof sketch. Consider automorphisms on these binary trees by "mirroring" at each vertex, switching the subsequent left and right subtree at the given vertex. Then we see these involutions have almost all size two orbits, except the trees with $2^k - 1$ many vertices that is symmetric at every vertex.

WEEK 2 MONDAY

11 Classes of combinatorial sequences (continued).

We have our universe of combinatorial sequences as follows:

Rational $\subset$ Algebraic $\subset$ D-finite $\subset$ ADE

and we have a class called $\mathbb{R}_+$-algebraic that is also within algebraic, whose intersection with rational are the $\mathbb{N}$-rationals.

12 What is not a combinatorial question?

Older generations would consider everything is combinatorial, but we look at the following instances where we might not consider them purely combinatorial.

(1) The conjecture: There are infinitely many Fibonacci numbers that are also prime. This is probably considered more of number theory.

(2) Observe that for A, B rational, then $-A, 1/A, A + B, AB$ are all rational; also we defined that if A, B are $\mathbb{N}$-rational, then so is $A + B, AB$, as well as $1/(1 - A)$ when $[1]A = 0$.

Now we in fact also have: For A, B algebraic, then $A + B$ and AB are both algebraic. This result, however, is really a consequence of algebra. (An analogy is if $\alpha, \beta \in \overline{\mathbb{Q}}$,then $\alpha + \beta, \alpha\beta \in \overline{\mathbb{Q}}$.) This may be why chapter 6 of Stanley's EC2 is hard to read.

13 D-finite.

We say $A \in K[[x]]$ is D-finite if there exists polynomials $c_0, c_1, \ldots, c_r \in K[x]$ (or $\mathbb{Z}[x]$) such that $c_0 A + c_1 A' + \cdots + c_r A^{(r)} = 0$.

Example 10. e^x, $\sin x$, and $\sum n! x^n$ are all D-finite. What about $\sum \binom{2n}{n} x^n$? Yes.

How do we get above definition and name? A generating function A being D-finite amounts to the fact that $\dim_{\mathbb{Z}[x]} \langle A, A', A'', \ldots \rangle < \infty$. This characterizes D-finite series:

Proposition 11. *A is D-finite $\iff$ $\dim_{Z[x]} \langle A, A', A'', \ldots \rangle < \infty$.*

How do we come up with above dimensionality condition? Note the Fibonacci numbers 1, 1, 2, 3, 5, 8, 13,... and the Lucas numbers 2, 1, 3, 4, 7, 11,... both satisfy the following form: $a_0 = p$, $a_1 = q$, $a_{n+2} = a_n + a_{n+1}$, for some p, q given. This space of all such sequences is 2-dimensional. Now, when we shift the index from the sequence $a_0, a_1, \ldots$ to $a_k, a_{k+1}, \ldots$, the shifted sequence is still in this space. As shifting index corresponds to taking derivative, we see that taking derivatives do not increase the dimensionality in this example. Hence we have the definition $\dim\langle A, A', \ldots\rangle < \infty$.

14 Why is algebraic $\subset$ D-finite?

We mentioned these inclusions in our universe of classes of generating functions, but why is algebraic $\subset$ D-finite? Take A an algebraic series, then we have $c_0A^r + \cdots c_r = 0$ for some $c_i \in \mathbb{Z}[t]$. Differentiate sufficiently then shows A is D-finite. (Note this is not as trivial, since when adding two sequences yield "cross terms".)

(Another question also is: Why does D-finite form a sum-K-algebra? This question is not obvious and is more algebra than combinatorics.)

Note D-finite is closed under addition and product.

15 P-recursive sequences.

There is in fact a characterization of D-finite series. We say a sequence (a_n) is P-recursive if there exists $c_i \in \mathbb{Z}[n]$ such that $c_0a_n + c_1a_{n-1} + \cdots + c_ra_{n-r} = 0$.

Example 12. Both $n!$ and $\binom{2n}{n}$ are P-recursive.

Proposition 13. *The sequence (a_n) is P-recursive $\iff$ $\sum a_nt^n$ is D-finite.*

(Why? ($\Longleftarrow$) Easy direction. ($\Longrightarrow$) Note falling factorial polynomials form a basis.)

16 Random walk on $\mathbb{Z}^d$.

Let a_n be the number of $0 \to 0$ walks of length n on $\mathbb{Z}^d$. When $d = 1$, $a_{2n} = \binom{2n}{n}$, $a_{2n-1} = 0$.

When $d = 2$ we have $a_{2n} = \sum_{k=0}^{n}\binom{2k}{k}\binom{2(n-k)}{n-k}\binom{2n}{2k}$, and $a_{2n-1} = 0$. "This is awful looking, but typical." Now, is this P-recursive? Yes!

Theorem 14. *Let $S \subset \mathbb{Z}^d$ with $|S| < \infty$, and a_n = number of length n walks on $\mathbb{Z}^d$ with steps in S. Then (a_n) is P-recursive.*

Note, already in $d = 2$, the associated generating function is not algebraic. (However, for $d = 1$, it is algebraic.)

WEEK 2 WEDNESDAY

17 More on D-finite generating functions.

We are still looking at classes of generating functions. Specifically we have algebraic $\subset$ D-finite. Recall that $A(t)$ is D-finite if there are polynomials $c_i \in \mathbb{Z}[t]$ such that $c_0A^{(r)} + c_{'}A^{(r-1)} + \cdots + c_rA^{(0)} = 0$.

We collect some results regarding D-finite generating functions.

Theorem 15. *A is D-finite if and only if $\dim_{\mathbb{Q}}\langle A, A', A'', \ldots\rangle < \infty$.*

Theorem 16. *A is algebraic $\Longrightarrow$ A is D-finite.*

Theorem 17. *A, B are D-finite, then (1) $A + B$ and (2) AB are both D-finite.*

Proof sketch. (1) Let $V = \langle A, A', A'', \ldots\rangle$ and $W = \langle B, B', B'', \ldots\rangle$, both finite dimensional spaces. Then note $\langle (A+B), (A+B)', (A+B)'', \ldots\rangle \subset V + W$. Hence $\dim\langle (A+B), (A+B)', (A+B)'', \ldots\rangle \leqslant \dim V + \dim W < \infty$. So we have $A + B$ is D-finite.

(2) Let $U = \langle AB, (AB)', \ldots\rangle$. Then $U \subset V \otimes W$, so we have $\dim U \leqslant (\dim V)(\dim W) < \infty$.

Theorem 18. *A is D-finite, B is algebraic, then $A(B(t))$ is D-finite, for $B(0) = 0$.*

For example: $e^{\frac{1}{\sqrt{1-4t}}}$ is D-finite, but $\sqrt{\sin t}$ is not D-finite.

Also recall from before:

Theorem 19. *A sequence (a_n) is P-recursive $\iff$ $A(t) = \sum a_nt^n$ is D-finite.*

Theorem 20. *If $A(t) = \sum a_nt^n$ and $B(t) = \sum b_nt^n$ are both D-finite, then their Hadamard product $(A * B)(t) = \sum a_nb_nt^n$ is also D-finite.*

For example: The sequence $a_n = \#\{\sigma \in \mathfrak{S}_n : \sigma^2 = 1\}$ counts the number of involutions in $\mathfrak{S}_n$.

Proposition 21. *(a_n) is P-recursive.*

Proof. Indeed, note $a_n = a_{n-1} + (n-1)a_{n-2}$, because $\sigma \in \mathfrak{S}_n$ fixes the symbol n or not. If σ fixes the symbol n, then σ restricts to a involution on $\mathfrak{S}_{n-1}$; if σ does not fix the symbol n, then $\sigma(n) \in [n-1]$. By deleting the 2-cycle $(n, \sigma(n))$ from σ we get an involution on $n-2$ symbols.

Continuing our example, we have $\sum a_n\binom{2n}{n}t^n$ is D-finite, as it is the Hadamard product of two D-finite series.

18 On diagonal series and D-finite series.

Consider $F \in \mathbb{Z}((x_1, x_2, \ldots, x_r))$, define the diagonal of F to be the generating function diag F where $(\operatorname{diag} F)(t) = \sum_{n=0}^{\infty} t^n [x_1^n x_2^n \cdots x_r^n]F$.

Theorem 22. *Let $P, Q \in \mathbb{Z}[x, y]$. Then diag $\frac{P(x,y)}{Q(x,y)}$ is algebraic.*

Theorem 23. *(Furstenberg) If $A(t)$ is algebraic, then there exists P, Q such that $A =$ diag $\frac{P}{Q}$.*

Theorem 24. *For $r \geqslant 1$ and $P, Q \in \mathbb{Z}[x_1, \ldots, x_r]$, we have diag $\frac{P}{Q}$ is D-finite.*

Remark 25. There are D-finite series that are not diagonals.

19 Connection to walks on $\mathbb{Z}^d$ are D-finite.

Suppose we have a walk process in $\mathbb{Z}^2$, with admissible steps $(1,1), (-1,1), (0,-1)$. We can consider the generating function

$$F = \frac{1}{1 - xy - x^{-1}y - y^{-1}}.$$

However, the coefficient $[x^i y^j]F$ does not make sense, as it is infinity (there are infinitely many walks from the origin to the point (i,j)). So let us modify it to

$$F = \frac{1}{1 - t(xy + x^{-1}y + xy^{-1})},$$

then $[x^i y^j t^n]F$ makes sense, this is the number of walks of length n to the point (i,j) from the origin.

But what can we say about $[x^i y^j t^n]F$? How about the diagonal $[x^n y^n t^n]F$, of which we know it is D-finite!

For instance the walk represented by $G = \frac{1}{1-(x+y+xy)}$ has $[x^i y^j]G$ well-defined (finite), whose diagonal $\operatorname{diag} G = \sum D_n t^n$ is given by the Delannoy sequence (D_n), which counts the number of walks to (n,n) using east, north, or northeast steps. We remark that $(\operatorname{diag} G)(t) = \frac{1}{\sqrt{1=6t+t^2}}$.

We further modify our F above to

$$F = \frac{1}{1 - txy(xy + x^{-1}y + y^{-1})},$$

then $[x^n y^n t^n]F$ counts the number of $(0,0) \to (0,0)$ walks of length n. Note this F is a rational function in variables t, x, y, and hence has diagonal a D-finite series.

This leads to the following theorem.

Theorem 26. *For a finite set $S \subset \mathbb{Z}^d$, let $a_n = a_n(S) = \#\ (0,0) \to (0,0)$ walks with steps in S of length n. Then (a_n) is P-recursive.*

Example 27. Let S be the set with unit north, south, east, west steps in $\mathbb{Z}^2$. Then in fact $a_{2n} = \binom{2n}{n}^2$, which is P-recursive.

Remark 28. Note above $a_{2n} = \binom{2n}{n}^2$ is almost like an accident. We can prove this by choosing new axes p and q in the $\pm 45°$ direction. This decouples the north, south, east, west steps in the new p^+, p^-, q^+, q^- steps, which happen to be independent (unlike the north, south, east, west steps are dependent!).

Also note that $\binom{2n}{n}^2 \sim \frac{1}{n}$, whose sum over n diverges. This shows "there are a lot of walks returning to origin", and suggests that the a walk on $\mathbb{Z}^d$ returns to origin with probability 1.

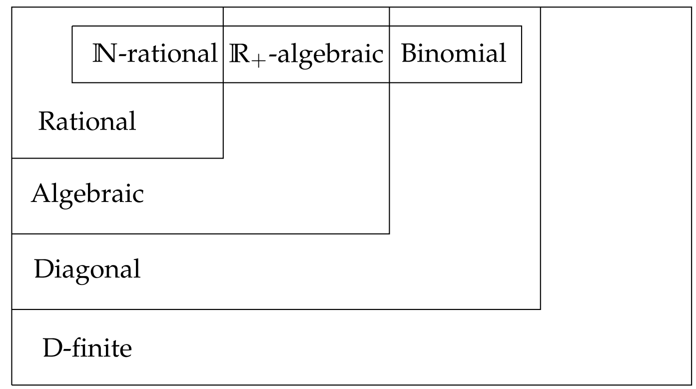
As for $d = 3$, we have $a_{2n} = \sum_{i+j+k=n} \binom{2i}{i}\binom{2j}{j}\binom{2k}{k}\binom{n}{i,j,k}$, also P-recursive. And here $a_{2n} \approx \binom{2n}{n}^3 \sim \frac{1}{\sqrt{n^3}}$, whose sum over n converges. Hence this suggest that a walk on $\mathbb{Z}^3$ returns to origin with probability 0.

Also observe here that a_{2n} is a sum of binomial coefficients. We will address these kinds of series later.

WEEK 2 FRIDAY

20 Classes of generating functions (continued).

So far, our universe of generating functions look like this:



We recall some results, and note some new ones.

Theorem 29. *Diagonal of $\frac{P(x,y)}{Q(x,y)}$ is algebraic, and the converse is true as well.*

Theorem 30. *Diagonal of $\frac{P(x_1,\ldots,x_r)}{Q(x_1,\ldots,x_r)}$ is D-finite.*

Theorem 31. *(Lipschitz 1988) For F a D-finite series, its diagonal $\operatorname{diag} F$ is also D-finite.*

Example 32. Let $a_n = \binom{2n}{n}$, we have $\sum a_n x^n = \frac{1}{\sqrt{1-4x}} = A(x)$ with $A = \operatorname{diag} \frac{1}{1-x-y}$. Note every algebraic is diagonal of a two variable rational.

Example 33. Let $B(t) = \sum \binom{2n}{n}^2 t^n$, which is D-finite, with $\binom{2n}{n}^2$ P-recursive, as $B = A * A$ (Hadamard product).

Theorem 34. *For A, B diagonal, we have $A * B$ also diagonal.*

Proof sketch. Let $A(t) = \sum_n [x^n y^n] F(x,y) t^n$ and $B(t) = \sum_n [u^n v^n] G(u,v) t^n$, we have $(A * B)(t) = \sum [u^n v^n x^n y^n] F(x,y) G(u,v) t^n$. Hence $A * B$ is diagonal.

21 When is a D-finite series also algebraic?

Here is a question: Is $B(t) = \sum \binom{2n}{n}^2 t^n$ algebraic? We know it is D-finite. And how about $E(t) = e^t$, is it algebraic? If $E(t)$ is algebraic, then specializing at an algebraic number, say 1, will give an algebraic number. But $E(1) = e \notin \overline{\mathbb{Q}}$. We should note that e is not algebraic is also not a simple question either (cf. Liouville theorem, which says we cannot approximate an algebraic number like $\frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ easily). This is hard already for E.

Back to $B(t) = \sum \binom{2n}{n}^2 t^n$. Note for (say) $t \leqslant 1/16$, we have

$$B(t) = \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} (2\cos\theta)^{2n} d\theta \right) t^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{1 - 16t\cos^2\theta}}.$$

Specializing at $t = 1/32$, we get $\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\theta}{\sqrt{1-\frac{\cos^2\theta}{2}}}$, which by look up in a table, we got that this is not an algebraic number.

This shows that the Hadamard product of two algebraic series need not be algebraic!

We remark that so far we are working over $\mathbb{Q}$. But over fields of positive characteristics the situation is very different.

Theorem 35. *If A, B both algebraic series, over a field of positive characteristic, then their Hadamard product $A * B$ is algebraic.*

Theorem 36. *For $A = \text{diag}\, \frac{P}{Q}$ with $P, Q \in K[x_1, \ldots, x_r]$ and K a positive characteristic field, then A is algebraic.*

22 A discussion of A algebraic $\Longleftarrow$ $A = \textbf{diag}\, \frac{P(x,y)}{Q(x,y)}$.

We illustrate how one can prove the diagonal of a two variable rational is algebraic.

Consider again $A(t) = \sum a_n t^n$ with $a_n = \binom{2n}{n}$. Then we have $A(t) = \text{diag}\, \frac{1}{1-x-y}$. Write $F(x,y) = \frac{1}{1-x-y}$. Using the knowledge of A is a diagonal of a two variable rational, we shall compute A directly, and showing A is algebraic.

Note that $(\text{diag}\, F)(z) = [s^0]F(s, z/s) = \frac{1}{2\pi i}\int_{|s|=\rho} \frac{F(s,z/s)}{s} ds = \sum_{\text{poles}} \text{Res}_s F(s,z,s)/s$, for some small enough ρ. (Recall Cauchy integral theorem and residue theorem here.)

And note that if $A(s)/B(s)$ has simple pole at s_0, and $A(s_0) \neq 0$, then $\text{Res}_{s_0} A/B = A(s_0)/B'(s_0)$.

So, $\text{diag}\, F = \frac{1}{2\pi i}\int \frac{ds}{s(1-s-z/s)} = \frac{1}{2\pi i}\int_{|s|=\rho>0} \frac{ds}{-z+s-s^2}$, with poles at $s = \frac{1}{2}(1 \pm \sqrt{1-4z})$. The only works is $s_0 = \frac{1}{2}(1 - \sqrt{1-4z})$. This gives $A(s_0)/B'(s_0) = \frac{1}{\sqrt{1-4z}}$.

This computational exercise illustrates that a diagonal of a two variable exercise can be shown to be algebraic.

23 Another kind: Binomial sums.

Consider $a_n = \sum_k \binom{2n}{n-k}\binom{n+k}{k}4^k$. Is this P-recursive? Yes, we can show that $\sum a_n t^n = (\sum b_n t^n)(\sum c_n t^n)$ for some b_n, c_n P-recursive sequences. In fact we have the following theorem:

Theorem 37. *Let $a_n = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_l} \binom{*}{*}\binom{*}{*}\cdots\binom{*}{*}$ where $*$ is something linear in n, k_i, then a_n is P-recursive. We call this kind a binomial sum.*

We shall see that in fact a binomial sum is a diagonal of something nice.

WEEK 3 MONDAY

24 Some examples and nonexamples of D-finite series.

Note that e^t, $\sin t$, $t^2 e^{\sqrt{1-t^2}}$, $\frac{1}{\sqrt{1-t^2}}$, e^{t^2} are all D-finite.

However, e^{e^t-1} is not D-finite, hence the sequence $(B_n/n!)$ is not P-recursive, where (B_n) are the Bell numbers. And consequently, (B_n) is also not P-recursive. But what kind of series are the Bell numbers (B_n)? We shall introduce a new class.

25 ADE series.

A series $A(t) = \sum a_n t^n$ is ADE (algebraic differential equation) if there exists a polynomial $Q \in \mathbb{Z}[t, x_0, x_1, \ldots, x_r]$ such that $Q(t, A, A', \ldots, A^{(r)}) = 0$.

Exercise 38. The Bell numbers $(B_n/n!)$ is an ADE series.

Example 39. Let $a_n = \#\{\sigma \in \mathfrak{S}_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}$, the number of alternating (zig-zag) permutations on n symbols. Consider $A(t) = \sum \frac{a_n}{n!} t^n$, we can show that $A = A'A''$ (when we take an alternating permutation, we can split it into two parts where 1 is). Hence A is an ADE series.

Here is a number theoretic example:

Theorem 40. *(Jacobi 1848) $\sum_{n=0}^{\infty} t^{n^2}$ is ADE.*

Note: $(\sum t^{n^2})^4$ gives number of ways to write an integer as sum of 4 square numbers, which is a result from number theory. This allows us to show $\sum t^{n^2}$ is ADE.

Conjecture 41. *$\sum t^{n^3}$ is NOT ADE.*

Theorem 42. *(Lipschitz,*,1980) $\sum t^{2^n}$ is NOT ADE.*

Theorem 43. *$\sum t^{n!}$ is NOT ADE.*

Proof idea: This is because the gap in $\{n!\}$ is too large for the series to satisfy an algebraic differential equation.

Theorem 44. *(Ramanujan, Jacobi) Let $p(n)$ =# of integer partitions of n. Then $P(t) = \sum p(n)t^n = \prod_{k=1}^{\infty} \frac{1}{1-t^k}$ is ADE.*

Note: $\frac{1}{P(t)} = \prod_{k=1}^{\infty}(1-t^k) = \sum_{m=-\infty}^{\infty}(-1)^m t^{m(3m-1)/2}$.

There are a lot about ADE we still do not know about. Our classification now look like this:

$\mathbb{N}$-rational
Rational
Algebraic
Bin. Sum/Diag.
D-finite
ADE

For now, let us go back down smaller, and look at some characterizations of these smaller classes of generating functions we mentioned before. In particular we look at $\mathbb{N}$-rational functions, and later the relation between binomial sums and diagonals.

26 Characterizations of $\mathbb{N}$-rational functions.

Previously we alluded to the following theorem regarding $\mathbb{N}$-rational functions:

Theorem 45. *Consider*

$$\mathcal{F} = \{A = \sum a_n t^n : a_n = \#(S \to T) \text{ paths in some finite graph } G\}$$
$$\mathcal{R} = \{A : A \text{ is } \mathbb{N}\text{-rational}\}.$$

Then $\mathcal{F} = \mathcal{R}$.

Recall that $\mathcal{R}$ is given by (i) $1, t \in \mathcal{R}$, (ii) $A, B \in \mathcal{R} \implies A + B, AB \in \mathcal{R}$, (iii) $[1]A = 0$ and $A \in \mathcal{R}$ implies $\frac{1}{1-A} \in \mathcal{R}$.

Now observe that $\mathcal{R} \subset \mathbb{N}[[t]]$ and $\mathcal{R} \subset \mathbb{Z}(t)$. Meaning that for $A(t) = \sum a_n t^n \in \mathcal{R}$, we have $a_n \in \mathbb{N}$ and that $A = \frac{P}{Q}$ with $P, Q \in \mathbb{Z}[t]$, $Q \neq 0$.

However, we have:

Theorem 46. $\mathcal{R} \subsetneq \mathbb{N}[[t]] \cap \mathbb{Z}(t)$.

Why do we have strict containment above? Well, a consequence from Berstel-Soittola's theorem says the sequence $a_n = 2 \cdot 5^n + (4-3i)^n + (4+3i)^n$ gives $A(t) = \frac{2}{1-5t} + \frac{1}{1-(4-3i)t} + \frac{1}{1-(4+3i)t}$ which is not $\mathbb{N}$-rational, as A fails some conditions for $\mathbb{N}$-rationals. However, it is clear that $a_n \in \mathbb{N}$ and that $A \in \mathbb{Z}(t)$, so $A \in \mathbb{N}[[t]] \cap \mathbb{Z}(t) - \mathcal{R}$. We record this theorem here:

Theorem 47. *(Berstel-Soittola, 1970s) For $A(t) = \sum a_n t^n = \frac{P(t)}{Q(t)}$, with $P, Q \in \mathbb{Z}[t]$ and $a_n \in \mathbb{N}$, we denote R_A to be the set of minimal poles of A (closet to origin). Then A is $\mathbb{N}$-rational if and only if for all $\rho \in R_A$ are of the form $\rho = \alpha e^{2\pi i(j/m)}$ with $j, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$.*

That is to say, the minimal poles of A need to have "rational angles" in the complex plane.

Hence, as $4 + 3i \neq \alpha e^{2\pi i(j/m)}$, since $\arctan(3/4) \notin \mathbb{Q}$, the above a_n sequence is not $\mathbb{N}$-rational.

Example 48. The sequence $b_n = 2(5)^n + (1-2i)^{2n} + (1+2i)^{2n}$ is $\mathbb{N}$-rational. Also for $c_n = 3^n - 1$, with $C(t) = \sum c_n t^n = \frac{1}{1-3t} - 1$, it is $\mathbb{N}$-rational as well.

WEEK 3 WEDNESDAY

27 Characterization of $\mathbb{N}$-rational functions, continued.

Last time we illustrated (B-S theorem):

Theorem 49. *Suppose $A(t) = \frac{P(t)}{Q(t)} = \sum a_n t^n$ with (i) $P, Q \in \mathbb{Z}[t]$ and (ii) $a_n \in \mathbb{N}$. Then A is $\mathbb{N}$-rational $\iff$ A satisfies some rational condition on the poles.*

Example 50. Consider $A(t) = \frac{1}{1-3t} - \frac{1}{1-t^2} = \frac{3t-t^2}{(1-t^2)(1-3t)}$, with $a_n = \begin{cases} 3^n & n \text{ odd} \\ 3^n - 1 & n \text{ even} \end{cases}$. We see that by the theorem, A is $\mathbb{N}$-rational, as A has minimal pole at $\rho = 1/3$, which has a rational angle in the complex plane.

Remark 51. Note, however, that it is not clear directly from the definition that A is $\mathbb{N}$-rational. This raises the following question: Given $P, Q \in \mathbb{Z}[t]$, is $A = P/Q$ an $\mathbb{N}$-rational function? What would be a procedure to decide that?

Remark 52. Also, as a corollary to above theorem, for any rational function $A \in \mathbb{Z}(t)$ with $[1]A = 1$, there exists some $\lambda > 0$ large enough such that $\widetilde{A} = \frac{1}{1-\lambda t} - A$ is $\mathbb{N}$-rational!

Let us turn to the other characterization of $\mathbb{N}$-rational functions:

Theorem 53. *Consider*

$$\mathcal{F} = \{A = \sum a_n t^n : a_n = \#(S \to T) \text{ paths in some finite graph } G\}$$
$$\mathcal{R} = \{A : A \text{ is } \mathbb{N}\text{-rational}\}.$$

Then $\mathcal{F} = \mathcal{R}$.

Proof sketch.

($\mathcal{F} \subset \mathcal{R}$) We look at graphs Γ of the kind: $\boxed{S} \to G \to \boxed{T}$ where S, T are source and target vertices, and G some intermediate parts (the other kinds can be written as a sum of this kind). Denote n to be the number of vertices not S, T in Γ.

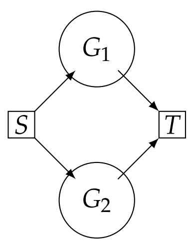
Observe that $\boxed{S} \to \boxed{T}$ has $A(t) = 1$ (no intermediates).

Suppose we have $\boxed{S} \to G \to \boxed{X} \to \boxed{T}$, where X is the vertex that goes to T, then enough to consider a = number of times a walk γ goes to X. If $a = 1$, then done. Otherwise, we consider paths from some β to X and then X to α, for $\alpha, \beta \in G$. So we get some $\sum_n \frac{1}{1-t^n} \sum_{\substack{\text{loops } C \\ |C|=n}} (*) = \frac{1}{1-\sum_C t^{|C|}}$, which shows $\mathbb{N}$-rationality. (?)

($\mathcal{R} \subset \mathcal{F}$) Since $0, 1, t \in \mathcal{F}$, we need show the following:

(i) $A, B \in \mathcal{F} \implies A + B, AB \in \mathcal{F}$.

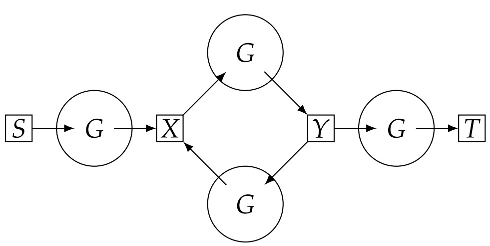
Indeed, consider a serial configuration $\boxed{S} \to G_1 \to \boxed{T} \to G_2 \to \boxed{Z}$, which gives the product AB. And a parallel configuration



gives the sum $A + B$.

(ii) $A \in \mathcal{F}$ and $[1]A = 0 \implies \frac{1}{1-A} \in \mathcal{F}$.

Note the condition $[1]A = 0$ is equivalent to $G \neq \varnothing$ for the graph $S \to [G] \to T$. Note we have the following graph in $\mathcal{F}$:



where the middle parallel part loops back on the bottom. This graph has series $\frac{A^3}{1-A^2}$.

Hence by noting $\frac{1}{1-A} = 1 + A + A^2 + \frac{A^3}{1-A^2} + A\frac{A^3}{1-A^2}$, we are done.

We note that for above to work we require S to be the source, with only outgoing edges, and T to be the target, with only ingoing edges.

Remark 54. Some history on this (Kleene and Schutzenberger). A language is "recognizable" if checking each word can be checked without backtracking. What kind of recognizable language are there? Turns out to be equivalent to paths $S \to [G] \to T$. (?)

WEEK 3 FRIDAY

28 Characterization of Diagonals.

Last time we showed $\mathcal{R}_1 = \{\mathbb{N}\text{-rationals}\}$ and $\mathcal{F}_1 = \{\text{path counting functions}\}$ are the same. We now try to extend this.

Define $\mathcal{F}_k = \{k\text{-counting functions in } k\text{-coloring of some graph } G\}$. What are these? For $G = (V, E)$ a directed graph, $S, T \in V$ with S, T source and target vertices, and $\eta : E \to [k]$ a k-coloring on the edges, denote $a_{n_1,\ldots,n_k}(G) = \# \, S \to T$ paths with n_i edges of color i, for each $i \in [k]$. Then $\sum a_{n_1,\ldots,n_k}(G) x_1^{n_1} \cdots x_k^{n_k}$ is the corresponding k-counting function. We have the following:

Theorem 55. *For fixed k, we have*

$$\mathbb{Z}\langle \mathcal{F}_k \rangle = \{\frac{P}{1-Q} : P, Q \in \mathbb{Z}[x_1, \ldots, x_k], [1]Q = 0\}.$$

Remark 56. Note this theorem implies for $k = 1$, we have $\frac{P(t)}{1-Q(t)} \in \mathcal{R}_1$, the $\mathbb{N}$-rational functions.

Proof idea for theorem: Let us denote this class of fractions $\mathcal{Q} = \{\frac{P}{1-Q} : P, Q \in \mathbb{Z}[x_1, \ldots, x_k], [1]Q = 0\}$. Note that we have $0, 1, x_1, \ldots x_k \in \mathcal{Q}$; for $A, B \in \mathcal{Q}$, we have $A \pm B, AB \in \mathcal{Q}$; and for $A \in \mathcal{Q}$ and $[1]A = 0$, we have $\frac{1}{1-A} \in \mathcal{Q}$. This shows $\mathbb{Z}\langle \mathcal{F}_k \rangle \subset \mathcal{Q}$.

To show $\mathcal{Q} \subset \mathbb{Z}\langle \mathcal{F}_k \rangle$, we use the idea of serialization (product AB) and parallelization (sum $A + B$) of these graphs as before.

Now we define the class $\mathcal{R}_k$ to be the $\mathbb{N}$-rational functions over k variables, it is such that (i) $0, 1, x_1, \ldots, x_k \in \mathcal{R}_k$, (ii) $A, B \in \mathcal{R}_k \implies A + B, AB \in \mathcal{R}_k$, and (iii) $A \in \mathcal{R}_k$, $[1]A = 0$ implies $\frac{1}{1-A} \in \mathcal{R}_k$.

Remark 57. Note we have $\mathcal{F}_k \subset \mathcal{R}_k$, but not necessarily the other way around.

Further, let us define $\mathcal{D}$ = diagonal of $\bigcup_{k=1}^{\infty} \mathcal{R}_k$. Also define $\mathcal{F} = \bigcup_{k=1}^{\infty} (\operatorname{diag} \mathcal{F}_k)$. Then:

Theorem 58. *We have $\mathcal{F} = \mathcal{D}$.*

To prove this, we in fact show they are both the same as another class: Binomial sums.

29 Binomial sums

Define $\mathcal{B}$ = the class of binomial sums. This is the collection of all series $B(t) = \sum b_n t^n$ where $b_n = \sum_{v\in\mathbb{Z}^k}\prod_{i=1}^r \binom{c_i\cdot v+d_in+e_i}{c_i'\cdot v+d_i'n+e_i'}$, where $c_i, c_i' \in \mathbb{Z}^k$, $d_i, d_i', e_i, e_i' \in \mathbb{Z}$. Note well that the terms in the binomial symbol are $\mathbb{Z}$-linear in v and n.

Example 59. The Delannoy numbers, $D_n = \sum_{k=0}^n \binom{n+k}{n-k}\binom{2k}{k} = [x^ny^n]\frac{1}{1-x-y-xy}$, is a binomial sum. And note that it is also a diagonal.

Example 60. The Apery numbers $A_n = \sum\sum\binom{n}{k}\binom{n+k}{k}\binom{k}{j}^3$. Note this is how Apery proved that $\zeta(3)$ is transcendental.

Example 61. For Lucas numbers $L_n = 2, 1, 3, 4, 5, ...$ and Fibonacci numbers $F_n = 1, 1, 2, 3, 5, 8, ...$, we have $L_n = F_n + F_{n-2}$, and $\sum L_n t^n = \frac{1+t^2}{1-t-t^2}$. Note, F_n is a binomial sum: Indeed, $F_n = \sum_k \binom{n-k}{k}$. But what about L_n? Here is a cool trick: Note $L_n = \sum_{k,i}\binom{n-k-2i}{k}\binom{1}{i} = \sum_k\binom{n-k}{k} + \binom{n-k-2}{k} = F_n + F_{n-2}$, since $\binom{1}{i}$ is nonzero when $i \in \{0, 1\}$. (Cool!) So L_n is also a binomial sum.

Finally, we aim to establish that:

Theorem 62. *$\mathcal{D} = \mathcal{F} = \mathcal{B}$.*

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30 Review from last time: On $\mathcal{B}, \mathcal{D}, \mathcal{F}$.

We want to establish the theorem that says $\mathcal{B} = \mathcal{D} = \mathcal{F}$, where

$$\mathcal{B} = \{\sum b_n t^n : b_n = \sum_{v\in\mathbb{Z}^d}\prod_{i=1}^r \binom{\alpha_i\cdot v + \alpha_i' n + \alpha_i''}{\beta_i \cdot v + \beta_i' n + \beta_i''}, \alpha_i, \beta_i \in \mathbb{Z}^d, \alpha_i', \alpha_i'', \beta_i', \beta_i'' \in \mathbb{Z}\}$$

$$\mathcal{D} = \bigcup_k (\text{diag. } \mathcal{R}_k),\ \text{where } \mathcal{R}_k \text{ are } \mathbb{N}\text{-rational functions in } k \text{ variables}$$

$$\mathcal{F} = \bigcup_k \{\text{path counting functions in finite graph } \Gamma \text{ with } k\text{-coloring on edges}\}.$$

Example 63. (1) Note $2^n = \sum_{i=0}^n \binom{n}{i} = \sum_{v\in\mathbb{Z}}\binom{n}{v}$, so the sequence $\{2^n\}$ gives a generating function in $\mathcal{B}$.

(2) Note $n^2 = \binom{n}{1}^2$, hence $\{n^2\}$ has a series in $\mathcal{B}$.

(3) Note Fibonacci numbers $F_n = \sum_{v\in\mathbb{Z}}\binom{n-v}{v}$, hence F_n gives a series in $\mathcal{B}$.

(4) Consider $b_n = \begin{cases} 1 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd}\end{cases}$. Is its series $\sum_n b_n t^n = \frac{1}{1-t^2}$ in $\mathcal{B}$? Indeed yes: $b_n = \sum_v \binom{n}{2v}\binom{2v}{n}$. Observe that $\binom{x}{y}\binom{y}{x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else}\end{cases}$.

(Note if we trust our hierarchy of our classes of generating functions, indeed we expect this to be in $\mathcal{B}$ as it is rational!)

Also, we shall take the convention that $\binom{-1}{0} = 1$.

(5) Consider $b_n = 2^n + 3^n$, note that

$$\begin{aligned} b_n &= \sum_{ij,k,l,m}\binom{n}{i}\binom{m}{j}\underbrace{\binom{1}{k}}_{\substack{\neq 0 \text{ if} \\ k=0,1}}\binom{m-k}{m}\binom{l+k-1}{l}\binom{i}{m+l}\binom{m+l}{i} \\ &= \underbrace{\sum\binom{n}{m}\binom{m}{j}}_{k=0} + \underbrace{\sum\binom{n}{l}}_{k=1} \\ &= 3^n + 2^n. \end{aligned}$$

Hence $2^n + 3^n$ has a series in $\mathcal{B}$.

We note the following:

Lemma 64. *$A, B \in \mathcal{D} \implies A + B, AB \in \mathcal{D}$. (Is this product here Hadamard?)*

Proof idea. For $A(t) = \operatorname{diag} F(x_1, \ldots, x_k)$, $B(t) = \operatorname{diag} G(y_1, \ldots, y_l)$ from $\mathcal{D}$, we have $A + B = \operatorname{diag}\left[\left(\prod\frac{1}{1-y_i}\right)F + \left(\prod\frac{1}{1-x_i}\right)G\right]$, as we have $a_n + b_n = [x_1^n\cdots x_k^n y_1^n\cdots y_l^n]\left[\left(\prod\frac{1}{1-y_i}\right)F + \left(\prod\frac{1}{1-x_i}\right)G\right]$. Note also we have $AB = \operatorname{diag} FG$.

Example 65. For $a_n = \binom{2n}{n}^2$, we showed before that $\binom{2n}{n}$ gives a series in $\mathcal{D}$, so $A(t) = \sum a_n t^n$ is also in $\mathcal{D}$. However, $A(t) \in \operatorname{diag}(\mathcal{R}_4)$ but $A(t) \notin \operatorname{diag}(\mathcal{R}_2)$, as A is not algebraic!

Note, asymptotically $\binom{2n}{n}^2 \sim \frac{4^{2n}}{\pi n}$ but this cannot be used to prove $\sum\binom{2n}{n}^2 t^n$ is not algebraic, just because π is not algebraic (a bizarre reasoning). Otherwise this would then apply to $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}$ yet $\sum\binom{2n}{n}t^n$ is algebraic.

31 Proof of $\mathcal{B} = \mathcal{D} = \mathcal{F}$.

The idea is to show $\mathcal{B} \subset \mathcal{D} \subset \mathcal{F} \subset \mathcal{B}$.

For $\mathcal{D} \subset \mathcal{F}$:

We have $\mathcal{R}_k \subset \mathcal{F}_k$, since we can show (i) $0, 1, x_1, \ldots, x_k \in \mathcal{F}_k$, (ii) $A, B \in \mathcal{R}_k \implies AB, A+B \in \mathcal{F}_k$, and (iii) $[1]A = 0, A \in \mathcal{R}_k \implies \frac{1}{1-A} \in \mathcal{F}_k$. Hence $\mathcal{D} \subset \mathcal{F}$.

For $\mathcal{B} \subset \mathcal{D}$:

(This is not "surprising" since $\mathcal{B}$ seems like a very small restrictive class.) Here we seek a way to write something from $\mathcal{B}$ as a diagonal.

Recall that $\sum \binom{2n}{n} t^n = \operatorname{diag} \frac{1}{1-x-y}$ and $\sum \binom{2n}{n}^2 t^n = \operatorname{diag} \left(\frac{1}{1-x-y}\right)\left(\frac{1}{1-u-v}\right)$. As it turns out that for $B(t) = \sum b_n t^n \in \mathcal{B}$, we can write $B = \operatorname{diag} \prod_i \frac{q_i(x_1,\ldots,x_n)}{1-p_i(x_1,\ldots,x_n)}$ for some polynomials q_i, p_i.

(Continued next time.)

WEEK 4 WEDNESDAY

32 $\mathcal{B} = \mathcal{D} = \mathcal{F}$ continued.

It remains to show $\mathcal{F} \subset \mathcal{B}$. The idea by direct enumeration, showing each walk and coloring corresponds to some binomial sum.

For a graph $G = (V, E)$ with $V = \{v_0, \ldots, v_k\}$, a cycle C in G is a sequence $C = (v_{i_1} \to v_{i_2} \to \cdots \to v_{i_l} \to v_{i_1})$. We say for such a cycle C,

(1) C is positive if $i_1 < \cdots < i_l$, and
(2) C is irreducible if it contains no smaller positive cycles.

Lemma 66. *For $G = (V, E)$ finite, then G has finitely many positive irreducible cycles.*

Proof idea. By contradiction. Suppose there exists a finite graph G that is a counterexample. Take such minimal G with infinitely many positive irreducible cycles. Then there exists infinitely many positive irreducible cycles that goes through all vertices of this minimal G. By positivity, such a cycle through all vertices must be of the form $C = (v_0 \to \cdots \to v_0)$. Then we can classify these positive irreducible cycles as two types: One that skips the vertex v_1 and the other starts from v_1 but never goes to v_0. However, both kinds are cycles in a smaller graph, hence a contradiction.

Suppose $\mathcal{C} = \{C_1, C_2, \ldots\}$ is the collection of all positive irreducible cycle in graph G, for any positive cycle C in G we can define a number $m_i(C)$ =# C_i in C as a subgraph. Or recursively, we can define $m_i(C) = \begin{cases} 1 + m_i(C - C_i) & \text{if } C \supsetneq C_i \\ 1 & \text{if } C = C_i \end{cases}$. Of course we need to make sure the following:

Lemma 67. *Above $m_i(C)$ is well-defined.*

Proof idea. By contradiction if it is instead dependent on the cycle decomposition. (?)

So, how does all of this lead to binomial sums?

Write $B(\alpha_1, \ldots, \alpha_r)$ =# positive cycles C in G $v_0 \to v_0$ such that $m_i(C) = \alpha_i$, and let k_{ij} = # times first vertex of C_i visits C_j, $1 \leqslant j < i$. Then we have

Lemma 68. $B(\alpha_1, \ldots, \alpha_r) = \prod_{i=1}^r \binom{[k_{i1}\alpha_1 + \cdots + k_{i,i-1}\alpha_{i-1}] + \alpha_i - 1}{\alpha_i}$.

Note above binomial coefficient is really a stars-and-bars expression, illustrating how a walk can go through different choices of the cycles.

WEEK 4 FRIDAY

(I missed this lecture due to proctor/grading)

33 Wilson's "cycle popping" algebra.

Something about an algorithm that takes an input G, output T a spanning tree in G. (??)

WEEK 5 MONDAY

34 Asymptotics of ADE and D-finite series.

For asymptotics of ADE, we know "nothing".

For asymptotics of D-finite we have the following "claim":

Conjecture 69. *Let $\{a_n\}$ be D-finite, integers, then $a_n \propto C(n!)^s \lambda^n n^\alpha (\log n)^\beta$ for $s \in \mathbb{Q}_+, \lambda \in \overline{\mathbb{Q}}, \alpha \in \mathbb{R}, \beta \in \mathbb{N}$.*

Firstly, what does the symbol $\propto$ mean in this case?

Example 70. For $a_n = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$, we cannot really talk about its asymptotics.

Note $A(t) = \sum a_n t^n = \frac{1}{1-t^2}$. Here $a_n = \frac{1}{2}(1)^n + \frac{1}{2}(-1)^n$, which is a finite sum of terms of the form $C\lambda^n$. So we write $a_n \propto C\lambda^n$. So $a_n \propto P$ means a_n is a finite sum of terms in the form P.

The asymptotics of some other classes are understood.

35 Asymptotics of rational functions and $\mathbb{N}$-rational functions.

Theorem 71. *For A a rational generating function, $A(t) = \frac{P(t)}{Q(t)} = \sum a_n t^n = \sum_i \frac{c_i}{(1-\rho_i t)^{k_i}}$, we have $a_n \propto C\lambda^n n^\alpha$ with $\alpha \in \mathbb{N}$, $\lambda \in \overline{\mathbb{Q}}$, and $c \in \overline{\mathbb{Q}}$.*

Example 72. Fibonacci $F_n \sim \frac{1}{\sqrt{5}}\phi^n$. Here we have $\sim$ instead of $\propto$ because the other term goes to 0.

Theorem 73. *For $\sum a_n t^n$ an $\mathbb{N}$-rational function, we have $a_n \propto C\lambda^n n^\alpha$ with $C \geqslant 0$ and $\lambda \geqslant 0$.*

36 Asymptotics of algebraic and $\mathbb{R}_+$-algebraic generating functions.

Theorem 74. *(Jugen 1931) For $\sum a_n t^n$ algebraic, we have $a_n \propto C\lambda^n n^\alpha$ with $\alpha \in \mathbb{Q}\backslash\{-1,-2,-3,\ldots\}$, $\lambda \in \overline{\mathbb{Q}}$, and $C \in \mathbb{R}$.*

Note as for algebraic $A(t)$ is also a diagonal of $\frac{P(x,y)}{Q(x,y)}$, this would imply $\lambda \in \overline{\mathbb{Q}}$ by thinking about the poles, which are algebraic.

Example 75. Note since $\binom{2n}{n}^2 \sim \frac{16^n}{\pi n} = \frac{1}{\pi}16^n n^{-1}$, above theorem implies that the generating function for $\binom{2n}{n}^2$ is not algebraic, since -1 is not an admissible power for n had it been algebraic. However, the theorem cannot conclude anything about $\binom{2n}{n}^3$ (although it is also not algebraic.)

Note that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi}}n^{-1/2}$, and $\sum \binom{2n}{n}t^n$ is algebraic.

Can we say something more about the asymptotic constant C for an algebraic series besides it being in $\mathbb{R}$? Not really:

Example 76. For Gessel walks using steps $(-1,1),(1,1),(0,-1)$, denote $g(n) =$ # of Gessel walks in the first quadrant.

Theorem 77. *(Gessel) $\{g(n)\}$ is algebraic, with $g(n) \sim \frac{2^{2/3}}{3\pi}\Gamma(1/3)16^n n^{-7/3}$.*

And the number $\frac{2^{2/3}}{3\pi}\Gamma(1/3)$ is not algebraic, due to the following specific result:

Theorem 78. *(Nesterenko, 1990s) $\pi, \Gamma(1/3), \log 2$ are algebraically independent.*

Another class of generating functions: $\mathbb{R}_+$-algebraic: We say $A(t)$ is $\mathbb{R}_+$-algebraic if $A(t) = P(t, A(t))$, where P is "well-defined positive".

Example 79. Catalan: $A(t) = 1 + tA^2$, $A(t) = \sum Cat(n)t^n$. Here $Cat(n) \sim \frac{1}{\sqrt{\pi}}4^n n^{-3/2}$ and is $\mathbb{R}_+$-algebraic.

Theorem 80. *If $A(t) = \sum a_n t^n$ is $\mathbb{R}_+$-algebraic, then $a_n \sim C\lambda^n n^\alpha$ where $\alpha = a/2^b$ (a dyadic rational). Moreover, $\alpha = (-1 - \frac{1}{2^b})$.*

Example 81. Gessel walks are not $\mathbb{R}_+$-algebraic.

37 Asymptotics of diagonal generating functions.

Theorem 82. *Suppose $A(t) = \sum a_n t^n$ is diagonal, $A = \text{diag}\,\frac{P(x_1,\ldots,x_k)}{1-Q(x_1,\ldots,x_k)}$, then $a_n \propto C\lambda^n n^\alpha(\log n)^\beta$, where $\alpha \in \mathbb{Q}$, $\lambda \in \overline{\mathbb{Q}}$, and $\beta \in \mathbb{N}$. Note: Here C can be crazy.*

Example 83. There exists a diagonal series $\sum a_n t^n$ with $a_n \sim \frac{\sqrt{\pi}}{\Gamma(\frac{5}{8})\Gamma(\frac{7}{8})}2^{7n}$. Note the crazy constant is from the evaluation of the hypergeometric ${}_2F_1(\frac{1}{4},\frac{3}{4};1,\frac{1}{2})$. This comes from $a_n = \sum_k \binom{4k}{k}\binom{3k}{k}128^{n-k}$, and as it is a binomial sum, it is a diagonal.

It is interesting to note we have the following conjecture:

Conjecture 84. *For any k, we have ${}_{k+1}F_k(\text{rational pt}) \neq e$.*

Also note this is the first emergence of the term $\log n$.

Example 85. For $a_n = \sum_{k=1}^n \binom{2k}{k}^2 16^{n-k}$, we have $a_n \sim \frac{16^n}{\pi}\log n$.

Example 86. The Catalan numbers give $A(t) = \sum Cat(n)t^n = \frac{1-\sqrt{1-4t}}{2y}$, which is algebraic and hence a diagonal. One can express this as $A(t) = \text{diag}\,\frac{1-x/y}{1-x-y}$ (try showing this by noting $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$). Another way is $A(t) = \text{diag}\,\frac{y(1-2xy-2xy^2)}{1-x-2xy-xy^2}$. However:

Conjecture 87. *$\frac{1-\sqrt{1-4t}}{2t} \notin \mathcal{F}$, that is, it is not a diagonal of a $\mathbb{N}$-rational function of some number of variables.*

WEEK 5 WEDNESDAY

38 Overview of asymptotics of the hierarchies of generating functions.

Ordering by inclusion, we have the following classes:

(1) $\mathbb{N}$-rationals
(2) Rationals
(3) $\mathbb{R}_+$-algebraic
(4) Algebraic
(5) Diagonal of $\mathbb{N}$-rationals = Binomial sums
(6) Diagonal of rationals = $\pm$Binomial sums
($6\frac{1}{2}$) G functions
(7) D-finite (with conjectural asymptotics)
(8) ADE (hard and not well understood)

39 Commentary on (5) vs (6).

Recall $A(t) = \sum a_n t^n$ a diagonal (either (5) or (6)), we have $a_n \propto C\lambda^n n^\alpha(\log n)^\beta$, for $C \in \mathbb{R}$, $\lambda \in \overline{\mathbb{Q}}$, $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$. Professor Pak and Scott made the following conjecture:

Conjecture 88. *(GP) If a_n is a binomial sum of type (5), then $\alpha = \frac{1}{2}\mathbb{Z}$.*

Why should it be believable?

Since $a_n = \sum_v \prod_{i=1}^r \binom{*}{*}$ with $*$ some linear expression in v and n, note that the mass of $\binom{*}{*}$ centers at the central coefficient $\binom{n}{n/2} \sim C\frac{2^n}{\sqrt{n}}$, this would suggest that $\alpha \in \frac{1}{2}\mathbb{Z}$.

(However there are difficulties in proving this: (1) If the center is at a corner of a polytope, and (2) issue of integrality.)

But $\alpha \in \frac{1}{2}\mathbb{Z}$ not required in (6) because things can cancel, giving other rational powers of n as leading terms. (?)

Now let $C_n = \frac{1}{n+1}\binom{2n}{n}$ the Catalan numbers. There is also:

Conjecture 89. *(GP) For $C(t) = \sum C_n t^n$, we have $C_n \neq$ diagonal of an $\mathbb{N}$-rational function. That is, C is not of class (5).*

Observe that $C(t)$ is of class (6) as $C_n = \binom{2n}{n} - \binom{2n}{n-1} \sim \frac{4^n}{\sqrt{\pi}n^{3/2}}$, the difference is what makes we wonder whether it can be class (5) or not.

They also made a stronger claim:

Conjecture 90. *(Pak) If $a_n \sim C_n$ as $n \to \infty$, then a_n is not of class (5).*

We do have, however, the following modularity and asymptotics results:

Theorem 91. *(1) For each m, there exists (a_n) of class (5) such that $(a_n) \equiv C_n$ mod m.*
(2) For each prime p, there exists (a_n) of class (5) such that $ord_p(a_n) = ord_p(C_n)$.

Proof. (1) Take $a_n = \binom{2n}{n} + (m-1)\binom{2n}{n-1}$, which is of class (5). This a_n does the job.

(2) Take $a_n = \binom{2n}{n} + (p^{2n}-1)\binom{2n}{n-1}$, which is of class (5). This a_n does the job.

(Note this is a generic technique of converting a type (6) to a type (5) so they agree modulo m.)

Theorem 92. *(1) There exists (a_n) of class (5) such that $a_n \sim \frac{3\sqrt{3}}{\pi}C_n$. ("...as they say in the late 90s, but wait, there's more!")*
(2) For each $\epsilon > 0$, there exists (a_n) of class (5) such that $(1-\epsilon)C_n < a_n < (1+\epsilon)C_n$, for n large enough.

Proof. (1) Take $a_n = \sum_k \binom{n}{3k}\binom{3k}{n}\binom{2k}{k}^3 \sim \binom{2(n/3)}{n/3}^3 \sim \left(\frac{4^{n/3}}{\sqrt{n/3}\sqrt{\pi}}\right)^3 \sim \frac{3\sqrt{3}}{\pi}C_n$. This a_n does the job.

(2) Can obtain this result from (1) by "multiplying appropriate things that remained a binomial sum".

40 G functions.

We say (a_n) gives $\sum a_n t^n$ a G function, if (a_n) is D-finite and that $|a_n| < C^n$ for some C. It is of class ($6\frac{1}{2}$), between $\pm$Binomial sum and D-finite. We have the following:

Theorem 93. *For (a_n) a G function, we have $a_n \propto C\lambda^n n^\alpha (\log n)^\beta$ with $\alpha \in \mathbb{Q}$, $\lambda \in \overline{\mathbb{Q}}$, and $\beta \in \mathbb{N}$.*

Conjecture 94. *(Garoufalidis) ($6\frac{1}{2}$)=(6).*

Theorem 95. *(Garoufalidis) Over the field $\mathbb{Q}$, above conjecture is false.*

WEEK 5 FRIDAY

41 Sequences over $\mathbb{Q}$ vs over $\mathbb{R}$.

We start with a conjecture:

Conjecture 96. *(Birkhoff-Trijitzinsky 1932) Suppose (a_n) is P-recursive (over $\mathbb{Q}$), then $a_n \propto C(n!)^s \lambda^n n^\alpha (\log n)^\beta \mu^{Q(n^{1/m})}$, for $s \in \mathbb{Q}$, $\mu, \lambda \in \overline{\mathbb{Q}}$, $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$, $m \in \mathbb{N}$ and $Q \in \mathbb{Q}[t]$. (Note they claimed this in a paper, but never proved.)*

Theorem 97. *For (a_n) P-recursive over $\mathbb{Q}$, $a_n \in \mathbb{N}$ and $a_n < C^n$, then $a_n \propto C\lambda^n n^\alpha (\log n)^\beta$ with $\lambda \in \overline{\mathbb{Q}}$, $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$.*

Corollary 98. *The partition function of integers $p(n)$ is not P-recursive, since $p(n) = e^{\Theta(\sqrt{n})}$.*

Note that $\{p(n)\}$ is ADE.

Example 99. Let a_n = # involutions in $\mathfrak{S}_n$, recall that $a_n = a_{n-1} + (n-1)a_{n-2}$, which means a_n is P-recursive. We have $A(t) = \sum a_n t^n = e^{t+t^2/2}$, and $a_n \sim \frac{1}{\sqrt{2}}\frac{1}{e^{1/4}}\left(\frac{n}{e}\right)^{n/2} e^{\sqrt{n}}$. Note a_n here is "weakly exponential".

Now let us consider sequences over $\mathbb{R}$.

Theorem 100. *(Gerhold, 2004) Let $a_n = \log n$, and $A(t) = \sum a_n t^n$. Then $A(t)$ is NOT D-finite.*

(Despite the fact that $a_n \sim C\lambda^n n^\alpha (\log n)^\beta$.)

Theorem 101. *If $A(t) = \sum a_n t^n$ is D-finite over $\mathbb{R}$, and $\lambda = 1$ with $a_n < C^n$, then $\beta = 0$.*

We look at some results regarding the sequence of prime numbers, (p_n).

Theorem 102. *(Flajolet-G-S, 2005) The primes (p_n) is NOT P-recursive.*

(It uses complex analysis, is there an elementary proof of this?)

Lemma 103. *We have $p_n \sim n \log n$.*

Lemma 104. *We have $p_n = n\log n + n \log\log n + O(n)$.*

Note the prime number theorem gives $\pi(n) = \#\{p_i < n\} = \int_2^n \frac{x}{\log x}dx + O(n)$. Consider the n-th harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Of course, H_n is P-recursive and $H_n \sim \log n$. Now the sequence $(p_n - nH_n) \sim n\log\log n$, but this shows nothing, as this is not an integer sequence.

Example 105. (Over $\mathbb{R}$) Is $a_n = \sqrt{n}$ P-recursive? No it is not.

Theorem 106. *(2004) $a_n = n^\alpha$ is P-recursive $\implies \alpha \in \mathbb{Z}$.*

Note $\dim_{\mathbb{Q}} \mathbb{Q}[\sqrt{2}, \sqrt{3}, \ldots] = \infty$, so $a_n = \sqrt{n}$ is not P-recursive.

Conjecture 107. *(G) Suppose (a_n) is P-recursive, with $a_n \in \mathbb{N}$ and $a_n < C^n$, then (a_n) is a diagonal of some $\frac{P}{1-Q}$, namely a_n is a $\pm$Binomial sum.*

Theorem 108. *Over $\mathbb{Q}$, above conjecture is false.*

42 More on G functions.

Let us say (a_n) is a G function if $a_n \in \mathbb{Q}$, $a_n < C^n$, and $a_n = \frac{\alpha_n}{\widetilde{\alpha}_n}$ is P-recursive, and the denominators $\widetilde{\alpha}_n$ satisfy $\gcd(\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_n) < C^n$.

Theorem 109. *There exists a G function which is not a diagonal of some $\frac{P}{1-Q}$.*

Proof. Consider $(2n+1)a_{n+2} - (7n+11)a_{n+1} + (2n+1)a_n = 0$, then $A(z) = \sum a_n z^n$ satisfies $z(z^2 - 7z + 2)A' + (z^2 - 4z - 3)A + z = 0$. Now, $z(z^2 - 7z + 2)$ has roots $\lambda \in 0, \frac{1}{4}(7 \pm \sqrt{33})\}$, which are algebraic. But this sequence has $\alpha = -1 \pm \frac{5}{2}\sqrt{\frac{3}{11}}$ which is not in $\mathbb{Q}$.

So, G functions are not the same as diagonals. (Conjecture: Over $\mathbb{Z}$, however, some thinks so.)

WEEK 6 MONDAY

43 Period numbers, hypergeometric numbers, constructible numbers.

Recall we have the theorem that says if (a_n) is P-recursive, with $a_n \in \mathbb{N}$, $a_n < c^n$, then $a_n \propto C\lambda^n n^\alpha (\log n)^\beta$ with $\lambda \in \overline{\mathbb{Q}}$, $\alpha \in \mathbb{Q}$, and $\beta \in \mathbb{N}$. We ask: What can this constant C be?

Number theorists study these number called period numbers $\mathcal{P} = \overline{\mathbb{Q}} \left\langle \int_a^b f(t)dt \right\rangle$ where $a, b \in \overline{\mathbb{Q}}$ and f algebraic. We have the following conjecture:

Conjecture 110. $C \in \mathcal{P}$.

Note that $\pi \in \mathcal{P}$ (as $\pi = 4\int_0^1 \sqrt{1-t^2}dt$), $\log 2 \in \mathcal{P}$.

Conjecture 111. *(Kontsevich, Zagier, 2000) $\frac{1}{\pi}, e \notin \mathcal{P}$.*

Conjecture 112. *The Catalan series $1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \notin \overline{\mathbb{Q}}$.*

Theorem 113. *(KZ) The hypergeometric number*

$$_{k+1}F_k(\alpha_1, \ldots, \alpha_{k+1}, \beta_1, \ldots, \beta_k, \gamma) \in \mathcal{P},$$

where $\alpha_i, \beta_i, \gamma \in \mathbb{Q}$.

A question we ask: Does there exist a constructible number that is not in $\overline{\mathbb{Q}} \langle F(...) \rangle$?

44 Walks on Cayley graphs.

Another question (Kontsevich): For G a subgroup of $SL(k, \mathbb{Z})$, $\langle S \rangle = G$, S a generating set of G, and let $a_n = \#$ of length n walks from $1 \to 1$ in the Cayley graph of G over S. Is (a_n) P-recursive? Answer: No!

Theorem 114. *For $G = \mathbb{Z}^m$, and S some generating set of G. Then (a_n) is P-recursive.*

Theorem 115. *(Haiman) $G = F_k$ (free group with k symbols), and S some generating set of F_k. Then (a_n) is algebraic.*

Note that $SL(2, \mathbb{Z}) \approx F_2$. Also we have Sarnov's subgroup $\left\langle \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle \approx F_2$, which has index 12 inside $SL(2, \mathbb{Z})$.

Note also if G a subgroup of F_k, then $G \approx F_l$ for some l.

Theorem 116. *For $G \approx \mathbb{Z} \ltimes \mathbb{Z}^2$ with generating set $S = S^{-1}$, $\langle S \rangle = G$, we have the sequence $a_n = a_n(G, S)$ not P-recursive.*

If we write $a_n = |S|^n p_n$, then we have:

Theorem 117. *(Varopoulos 1986, Alexopoulos 1992) $p_n \approx C\rho^{-n^{1/3}}$ (*).*

And hence implies $a_n \sim |S|^n \rho^{-n^{1/3}}$, which is not allowed for P-recursive sequences (the exponent $n^{1/3}$), and therefore answering Kontsevich's question.

(*) Here G is a linear and solvable group, and hence polycyclic bifinite. ($G_i / G_{i+1} \approx \mathbb{Z}$ is polycyclic bifinite?)

Example 118. $G = \mathbb{Z}^d \ltimes \mathbb{Z}_2^{\mathbb{Z}^d}$, for $d = 1$ we have the lamp lighter group. This is not a linear group, but is solvable, not polycyclic.

We ask the question: Does there exists a group G, $\langle S_1 \rangle = \langle S_2 \rangle = G$ such that $a_n(G, S_1)$ is P-recursive and $a_n(G, S_2)$ is not P-recursive?

Proposition 119. *For $G = \langle S_1 \rangle = \langle S_2 \rangle$, then $a_n(G, S_1) < Ca_{\alpha^n}(G, S_2) + D$, where α is "max length".*

So if $\rho^{n^{1/3}} < \frac{a_n(G, S_1)}{|S_1|^n} < \eta^{n^{1/3}}$, the same for $a_n(G, S_2)$ with similar forms of bounds.

WEEK 6 WEDNESDAY

45 Combinatorics of words, mod 2 sequences, and word complexity.

Recall we have the problem: Let G a finitely generated group with finite set S such that $\langle S \rangle = G$, and $a_n = a_n(G) = \#\{1 = s_{i_1} s_{i_2} \cdots s_{i_n} : s_{i_j} \in S\}$. Is $a_n(G)$ P-recursive?

Last time we have: Yes for free group F_k and $\mathbb{Z}^m$, but no for the lamp lighter group $\mathbb{Z} \rtimes \mathbb{Z}^2$.

We now look at how P-recursive sequences a_n relate to its associated binary word (or its mod 2 sequence) α_n:

Lemma 120. *Suppose (a_n) is P-recursive, and let $\overline{\alpha} = (\alpha_1, \alpha_2, \dots) \in \{0,1\}^\infty$ with $\alpha_n \equiv a_n \mod 2$. Then there exists a binary word $w = (w_1, \dots, w_l)$ such that w is NOT a subword of $\overline{\alpha}$.*

Example 121. Consider Fibonacci numbers (F_n), which is P-recursive, whose mod 2 sequence is $(1,1,0,1,1,0,1,1,0,...)$. Note we do not have the subword $w = (00)$ or $w = (111)$.

Example 122. Consider $\binom{2n}{n}$, which has mod 2 sequence $(1,0,0,0,...)$. Hence does not have the subword $w = (11)$. We can also see this by examining the mod 2 version of the Pascal triangle, and look at the central column which gives the central binomial coefficients. Here the mod 2 Pascal triangle looks like a Sierpinski's triangle.

For the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$, it does not have the subword (111). We can see this by recalling the following result:

Theorem 123. *(Kummer) The Catalan numbers* $C_n \mod 2 = \begin{cases} 1 & n = 2^k - 1 \\ 0 & \text{otherwise} \end{cases}$.

Example 124. But what about the P-recursive sequence $a_n = \binom{3n}{n}$? Which subword does its mod 2 sequence avoid? This is not so clear!

Example 125. Let $a_n = \#$ of involutions in $\mathfrak{S}_n$. Note $a_{n+1} = a_n + na_{n-1}$, so (a_n) is P-recursive. Observe that its mod 2 sequence goes as $(1,0,0,0,...)$. Or note $a_n \equiv n! \mod 2$.

Now what is the idea behind the proof of above lemma?

Note for (a_n) P-recursive, we have some $k+1$ polynomials $p_0, p_1, \dots, p_k \in \mathbb{Z}[n]$ such that $(a_n, a_{n-1}, \dots, a_{n-k}) \cdot (p_0(n), \dots, p_k(n)) = 0$. Now, look at the map $F_n : (a_{n-k}, \dots, a_{n-1}) \mapsto (a_{n-k+1}, \dots, a_{n-1}, a_n)$, whose mod 2 version maps $(\alpha_{n-k}, \dots, \alpha_n) \mapsto (\alpha_{n-k+1}, \dots, \alpha_n)$. But now consider the map F_{n+2^N}, at some point it becomes stabilized and periodic. Then take that whole period and switch with the following term to produce a subword that it would avoid.

Let us look at some application of this lemma:

Example 126. Consider the binary sequence

$$\overline{\alpha} = (0\,1\,00\,01\,10\,11\,000\,001\,010\,011\,100\,101\,110\,111...),$$

which contains all possible finite binary subwords. Hence we have a corollary: For any sequence (a_n) such that $a_n \equiv \alpha_n \mod 2$ then (a_n) is not P-recursive.

Example 127. Recall we have the theorem that says (p_n) is not P-recursive, where p_n is the n-th odd prime. Now define $\alpha_n = \frac{p_n - 1}{2} \mod 2$, we have

$$\begin{array}{cccccccc} p_n & 3 & 5 & 7 & 11 & 13 & 17 & 19 & \cdots \\ \alpha_n & 1 & 0 & 1 & 1 & 0 & 0 & 1 & \end{array}$$

and we have the following claim about this binary sequence $\overline{\alpha} = (\alpha_n)$:

Conjecture 128. *$\overline{\alpha} = (\alpha_n)$ contains all subwords $w \in \{0,1\}^*$.*

We do have, however:

Theorem 129. *For each k, $w = (1 \cdots 1)_k$ and $v = (0 \cdots 0)_k$ are subwords of $\overline{\alpha} = (\alpha_n)$.*

A weaker claim is this:

Conjecture 130. *Let $c_k(\overline{\alpha})$ be the number of subwords of length k in $\overline{\alpha}$ (word complexity), then $c_k > \epsilon 2^k$ for every $\epsilon > 0$.*

Proposition 131. *$c_k(\overline{\alpha}) = 2^k$ or $< (2-\delta)^k$ for some δ. (?)*

What about for integer partitions?

Conjecture 132. *Let $p(n)$ be the number integer partitions of n, then its mod 2 sequence $\overline{\alpha}$ is such that $c_k(\overline{\alpha}) = 2^k$ (for larger enough k?)*

Exercise 133. The mod 2 sequence $(p(n) \mod 2)$ has infinitely many 0's and infinitely many 1's. (Idea: Use Euler's pentagonal's theorem.)

Conjecture 134. *We have $\frac{\#\{n<N: p(n) \equiv 0 \mod 2\}}{N} \to \frac{1}{2}$ as $N \to \infty$.*

Theorem 135. $\Theta(\sqrt{N}) < \#\{n < N : p(n) \equiv 0 \mod 2\} < N - \Theta(\sqrt{N})$.

WEEK 6 FRIDAY

46 Non-P-recursive sequences.

Recall from last time:

Lemma 136. *For (a_n) P-recursive and $\overline{\alpha} = (\alpha_1, \alpha_2, \dots)$ its mod 2 sequence, $\alpha_k \equiv a_k \mod 2$ and $\alpha_k \in \{0,1\}$. Then there exists $w \in \{0,1\}^*$ such that w is not a subword of $\overline{\alpha}$.*

And recall we define the word complexity $c(\overline{\alpha}, k) = \#\{w \in \{0,1\}^* : |w| = k, w \subset \overline{\alpha}\}$.

Then note that above lemma $\iff$ there exists k such that $c(\overline{\alpha}, k) < 2^k$ $\iff$ there exists $\delta > 0$ such that $c(\overline{\alpha}, k) < (2-\delta)^k$, for $\overline{\alpha}$ the mod 2 sequence of a P-recursive sequence.

Conjecture 137. *For P-recursive sequences and its mod 2 sequence $\overline{\alpha}$, we have $c(\overline{\alpha}, k) = O(k)$ (or $= O(k^3)$).*

Note for Catalan numbers we have $c(\overline{\alpha}, k)$ at about $O(2k)$.

Example 138. The Thue-Morse word w is defined the be the limit word $w = \lim w_k$ where $w_0 = 0$ and $w_{k+1} = w_k\overline{w}_k$. So it goes like $w = 011010010010110....$ We know the following:

Theorem 139. *For the Thue-Morse word w, we have $c(w, k) \leqslant 4k$.*

Theorem 140. *There does not exists any word $a \in \{0,1\}^*$ such that $a^3 \in w$.*

So, how does one go about constructing a non-P-recursive sequence? Well, by above lemma, if we have a sequence whose mod 2 sequence contains all possible finite binary words, then it cannot be P-recursive. An example of such mod 2 sequence is $\overline{\alpha} = (0\,1\,00\,01\,10\,11\,000\,001\,010\,011\,100\,101\,110\,111...)$, the sequence that contains all possible integers in binary. So we try to construct a sequence that is congruent to it.

We will consider Turing machines, which are equivalent to a finite automatons with 2 stacks (?). Take a graph Γ with a source and target vertex S and T, whose edges (ij) is weighted by $w_{ij} \in F_k \times F_l$, elements of the product of two fixed free groups. Let $a_n(\Gamma)$ =# balanced walks γ from $S \to T$ of length n. The walk γ is balanced if $\prod_{ij\in\gamma} w_{ij} = 1_{F_k \times F_l}$.

Remark 141. Given a product is in F_k, it is easy to tell if it is $1 \in F_k$, and in fact:

Theorem 142. *Deciding whether word$(x_1, \ldots, x_k) = F_n$ is $O(k)$ time.*

Now if a unary string is accepted by this Turing machine, then there exists a path from S to T that is balanced (?)

Lemma 143. *For the "all possible integers" binary sequence $\overline{\alpha} = (0\,1\,00\,01\,10\,11\,000\,001\,010\,011\,100\,101\,110\,111...)$, there exists Γ such that $a_n(\Gamma) = \alpha_n \mod 2$ for each n.*

Proof idea. This is a computable sequence, so there is a Turing machine that computes this sequence, so by equivalence, there is such a Γ. (In fact, there exists a Γ with 8 vertices that realize this!)

[We then discussed how a group G with generating set S can be constructed to have its Cayley graph equal to this graph Γ?]

Lemma 144. *There exists a group G with finite generating set S such that $\{a_n(G, S)\}$ is not P-recursive.*

Note also, there exists subgroup $G \subset F_k \times F_l$ such that with finite generating set S we get a P-recursive sequence and with another finite generating set S' we do not.

WEEK 7 MONDAY

(Lecture given by Damir.)

47 Binary partitions, and how to compute them fast.

Denote $q(n)$ = # of binary partitions of n, where $\sum q(n)t^n = \prod_{k=0}^{\infty} \frac{1}{1-t^{2^k}}$. For example $q(5) = 4$. This sequence goes as $(q(n)) = (1, 1, 2, 2, 4, 4, 6, 6, 10, 10, 14, 14, ...)$. Note we have the following recurrence:

Proposition 145. $q(0) = q(1) = 1, q(2n+1) = q(2n), q(2n) = q(2n-1) + q(n)$. *Also, $q(2n) = q(0) + \cdots + q(n)$.*

Using above proposition we can figure out the asymptotics of $q(n)$:

Theorem 146. $\log q(n) \sim C(\log n)^2$.

Proof sketch. $q(2n) = q(0) + \cdots + q(n) \leqslant nq(n)$, so $q(2^n) \leqslant 2^{n-1}q(2^{n-1}) \leqslant 2^{n-1}2^{n-2} \cdots 2^1 2^0 = 2^{\frac{n(n-1)}{2}}$. This gives upper bound. Now note that $q(4n) = q(0) + q(1) + \cdots + q(2n) \geqslant 2nq(n)$, so as above, we can get lower bound.

Note as a corollary:

Corollary 147. *$q(n) = e^{C(\log n)^2} \sim n^{\log n}$, and hence not P-recursive.*

We have the following asymptotics result:

Theorem 148. *(Mahler, deBruijn 1930-40s) We have* $\log q(n) \sim c_1(\log \frac{n}{\log n})^2 + c_2 \log n + c_3 \log\log n + O(1)$.

Now consider $N = k_1 2^n + k_2 2^{n-1} + \cdots + k_{n+1}2^0$, what is $q(N)$ = # ways $(k_1, \ldots, k_{n+1})$ that holds? How do we compute this efficiently? That is, computing $q(N)$ is $O(N)$, but what about computing $q(2^n)$? We convert this to a another equivalent sequence, Cayley compositions.

48 Cayley compositions.

For $N = k_1 2^n + k_2 2^{n-1} + \cdots + k_{n+1}2^0$, $N < 2^{n+1}$, observe that we have $0 \leqslant k_1 \leqslant 1$. If $k_1 = 0$, then $k_2 \in \{0, 1, 2, 3\}$. If $k_1 = 1$ then $k_2 \in \{0, 1\}$, and so on.

Introduce new variables: Write $k_1 = 2 - a_1$, so $1 \leqslant a_1 \leqslant 2$; $k_2 = 2a_1 - a_2$, so $1 \leqslant a_2 \leqslant 4$ if $a_1 = 2$, or $1 \leqslant a_2 \leqslant 2$ if $a_1 = 1$. Define $k_3 = 2a_2 - a_3$, and so on. This is an affine transformation. So now $N = (2-a_1)2^n + (2a_1 - a_2)2^{n-1} + (2a_2 - a_3)2^{n-2} + \cdots + (2a_n - a_{n+1})2^0$. Here we have $1 \leqslant a_1 \leqslant 2$ and $1 \leqslant a_{i+1} \leqslant 2a_i$. This gives a bijection between the $k's$ sequences and the $a's$ sequences. Note $N = 2^{n+1} - a_{n+1}$. Note the (a_n) as defined

$$\begin{aligned} 1 \leqslant a_1 &\leqslant 2 \\ 1 \leqslant a_{i+1} &\leqslant 2a_i \\ a_{n+1} &= 2^{n+1} - N \end{aligned}$$

is called a Cayley composition of N.

With above affine transformation showed to be a bijection, we have indeed $q(N)$ many such Cayley compositions of N.

To count this, we note that these Cayley composition can be arranged in a tree T, where reading down from the root to the k-th level gives a particular Cayley composition of length k.

```
     /                \
   1                   2
 /   \             /  /     \     \
 1    2         1    2     3      4
12 1234      12 1234 1...6 1...8
```

(This is a step I didn't figure out this past summer.) Now let us define $p_n(x)$ = the number of n-th elements of x in T. Then note $p_n(1)$ is what we wanted. Note we have $p_1(x) = 2x$ and $p_{n+1}(x) = q(0) + q(1) + \cdots + q(2^{n+1} - 1)$.

So what is $p_n(x) = \sum_{j=1}^{2x} p_{n-1}(j)$? Note $p_2(x) = 2x(2x+1)$, and $p_n(x)$ are polynomials in x of degree n. To get $p_n(x)$, as it is a polynomial, just need $n+1$ points to determine it. This takes about $\sim \log N$. Then we compute $p_n(1)$ for our answer.

(Note $p_n(x+1) - p_n(x) = p_{n-1}(2x+1) + p_{n-1}(2x+2)$, which we store in a table of size $n \times (2n+1)$.)

WEEK 7 WEDNESDAY

(Lecture given by Damir.)

49 Faulhaber's formula and its generalization.

Consider $1^m + 2^m + \cdots + (N-1)^m = \sum_{i=1}^{N-1} i^m = \frac{1}{m+1} \sum_{i=0}^{m} B_i \binom{m+1}{i} N^{m+1-i}$, where B_i are the Bernoulli's numbers (in $\mathbb{Q}$). This is Faulhaber's formula and it yields a polynomial in N.

Example 149. We note some computed results for odd powers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = t(n) = t$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2 = t^2$$
$$\sum_{i=1}^{n} i^5 = \frac{6t^3 - t^2}{3}$$
$$\sum_{i=1}^{n} i^7 = \frac{12t^4 - 8t^3 + 2t^2}{6},$$

where $t(n)$ denotes the n-th triangular number. This observation leads to the following result:

Theorem 150. *(Faulhaber 16xx) For $t(n) = \frac{n(n+1)}{2}$, we have*

$$\sum_{i=1}^{n} i^{2m+1} = poly_m(t(n))$$
$$\sum_{i=1}^{n} i^{2m} = poly_m(t(n)).$$

(This is proved by Jacobi.)

Now, fix m, k, and consider $\sum_{i=0}^{N-1} \binom{i}{k}^m = \text{poly}_{mk+1}(N) =: f_{k,m}(N)$. Then we have

Proposition 151. *For some $Q_{k,m} \in \mathbb{Q}[x]$,*

$$f_{k,m}(x) = \begin{cases} \binom{x}{k+1}^2 Q_{k,m}((2x+k-2)^2) & m, k \text{ odd}, m > 1 \\ \binom{x}{k+1}(2x+k-2) Q_{k,m}((2x+k-2)^2) & k \text{ odd}, m \text{ even} \\ \binom{x}{k+1} Q_{k,m}((2x+k-2)^2) & \text{otherwise.} \end{cases}$$

Note this is a generalization of Faulhaber's result, which we get when we set $k = 1$.

When $k = 2$, we get $\sum \binom{i}{2}^m = \binom{x}{3} Q(x^2)$, which is an odd polynomial, and in fact has the form $\sum \lambda_j x^{2j+1}$ (?).

Let us now denote

$$\sum_{i=0}^{N} i^{2m-1} = \frac{1}{2m} \sum_{i=0}^{m-1} F_i(m) (N(N+1))^{m-i}$$
$$\sum_{i=0}^{N} i^{2m} = (N - \frac{1}{2}) \sum_{i=0}^{m} \widetilde{F}_i(m) (N(N+1))^{m-i}.$$

Here we have $\widetilde{F}_i(m) = \frac{m+1-i}{(2m+1)(m+1)} F_i(m+1)$. Denote also $f_{2,m}(N) = \sum_{i=1}^{N} (\frac{i(i+1)}{2})^m = \frac{1}{2^m} \sum_{i=1}^{m} \overline{F}_i(m) N^{2m-2i+1}$. We then have the following reciprocity result:

Theorem 152. *(Dzhumadil'daev, Yeliussizov) We have reciprocity*

$$\overline{F}_i(m+i-1) = (-1)^i \widetilde{F}_i(-m).$$

.

50 Zeta function and its generalization.

Consider $\zeta_k(m) = \sum_{i=k}^{\infty} \frac{1}{\binom{i}{k}^m}$. Note when $k = 1$ then we get the usual zeta function $\zeta_1(m) = \zeta(m) = \sum_{i=1}^{\infty} \frac{1}{i^m}$. Then:

Proposition 153. *We have* $\zeta_k(m) \in \begin{cases} \mathbb{Q} + \mathbb{Q}\zeta(2) + \cdots + \mathbb{Q}\zeta(2l) & \text{if } km \text{ even} \\ \mathbb{Q} + \mathbb{Q}\zeta(3) + \cdots + \mathbb{Q}\zeta(2l+1) & \text{if } km > 1 \text{ odd.} \end{cases}$

We record some assorted results regarding the usual zeta function.

Proposition 154. *We have*
(1) (Euler) $\zeta(2k) = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}$, *where* B_{2k} *is a Bernoulli number.*
(2) (Apery) $\zeta(3)$ *is irrational. Note* $\zeta(2k+1)$ *is not as well understood.*
(3) (Rivoal 2000) There are infinitely many in the list $\{\zeta(5), \zeta(7), \zeta(9), \ldots, \zeta(2k+1), \ldots\}$ *that are irrational.*
(4) At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ *is irrational. (Hard!)*

How did Apery prove that $\zeta(3)$ is irrational? The idea: Take a P-recursive sequence u_n such that $(n+1)^3 u_{n+1} - (34n^3 + 51n^2 + 27n + 5)u_n + n^3 u_{n-1} = 0$.

With $u_0 = 1$ and $u_1 = 5$ as initial conditions, we have $u_n \in \mathbb{Z}$. Then these gives what are called the Apery numbers: $u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$.

With different initial conditions $v_0 = 0$ and $v_1 = 6$ we get $(\text{lcm}(1, \ldots, n))^3 v_n \in \mathbb{Z}$.

Now look at $|u_n\zeta(3) - v_n| < \alpha^n$ to show $\zeta(3)$ is irrational.

Nesterenko's approach: Take $F(k) = \left[\frac{(k-1)\cdots(k-n)}{k(k+1)\cdots(k+n)}\right]^2$ and $I_n = \sum_{i=1}^{\infty} F'(i) = A_n\zeta(3) + B_n$. Here $A_n \in \mathbb{Z}$ and $B_n \in \mathbb{Q}$, and $[\text{lcm}(1, \ldots, n)]^3 B_n \in \mathbb{Z}$. But $I_n \sim \frac{-\pi^{3/2}2^{3/4}}{n^{3/2}}(\sqrt{2}-1)^{4n+2}(1 + o(1))$. So the ratio is < 1, but this is a contradiction as positive integers cannot be < 1.

(Friday is Veteran's day.)

WEEK 8 MONDAY

(I missed this lecture due to proctor/grading)

51 On ADE generating functions: Partitions and Eisenstein series.

Recall that a sequence (a_n) is ADE if $A(t) = \sum a_n t^n$ satisfies $Q(t, A, A', \ldots, A^{(r)}) = 0$ for some $Q \in \mathbb{Z}[t, z_0, z_1, \ldots, z_r]$.

Example 155. For $a_n = \#\{\sigma \in \mathfrak{S}_n : \sigma(1) < \sigma(2) > \sigma(3) < \cdots\}$, $A(t) = \sum a_n t^n$ satisfies $A = 1 + A' + A''$.

Theorem 156. *(Jacobi 1848)* $\sum t^{n^2}$ *is ADE.*

(Real Jacobi's theorem: $(A^2 A''' - 15AA'A'')^3 + \cdots = 0$. *Something like that.)*

Theorem 157. *For* $p(n)$ *the number of integer partitions of* n*, we have the sequence* $(p(n))$ *is ADE.*

Theorem 158. *(Ramanujan) We have divisibility* $5|p(5n-1)$.

We define the Eisenstein series $E_k(z) = \frac{1}{2}\sum_{\substack{c,d \in \mathbb{Z} \\ \gcd(c,d)=1}} \frac{1}{(cz+d)^k}$ when k is even, and $E_k(z) = 0$ when k is odd. Write $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ for $q = e^{2\pi i z}$, and write $P(t) = \prod_{n=1}^{\infty}\frac{1}{1-t^n} = \sum p(n)t^n$. We have:

Theorem 159. $\Delta(z) = \frac{1}{1728}(E_4(z)^3 - E_6(z)^2)$.

So what this means is: If E_4 and E_6 are ADE, then Δ is ADE, and hence P is ADE.

Lemma 160. *(Ramanujan)* $E_2' = \frac{E_2^2 - E_4}{12}$, $E_4' = \frac{E_2E_4 - E_6}{3}$, *and* $E_6' = \frac{E_2E_6 - E_4^2}{2}$.

Corollary 161. E_2, E_4, *and* E_6 *are ADE.*

Proposition 162. *If* (a_n) *is ADE, then* a_n *can be computed in poly(n) time.*

Example 163. By pentagonal theorem, we have $p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots$. So $p(n)$ can be computed in $O(n^2)$ time.

Theorem 164. $\#\{n < N : p(n) \equiv 1 \mod 2\} = \Omega(\sqrt{N})$.

(Best known is $\Omega(\sqrt{N}\log N)$.*)*

Lemma 165. *(G-P) Suppose* (a_n) *is an integer sequence,* n_k *denotes k-th non-zero s.t.*
(1) for every b, c *there exists* k *such that* $n_k = b \mod 2^c$,
(2) $n_k/n_{k+1} \to 0$ *as* $k \to \infty$.
Then (a_n) *is not ADE.*

Corollary 166. $\sum t^{k!+k}$ *is not ADE.*

WEEK 8 WEDNESDAY

52 On Non-ADE generating functions, computability of sequences, and complexity.

Today we will look at sequences that are not ADE and how it relates to computability of these sequences. We start with this question:

Question: Given sequence (a_n), can we compute its terms in time poly(n)?

This is Wilf's question in the 1980s. (I think he may have regarded "an answer" as something that can be computed in polynomial time.)

Proposition 167. *If* (a_n) *is ADE, then* a_n *is computable in poly(n) time.*

Example 168. The number of integer partitions of n, $p(n)$, can be computed in $O(n^2)$. Can it be done in linear time?

Question: Can $a_n = \#$ unlabeled graphs on n vertices, be computed in polynomial time? This is unknown and the conjecture is no. Here $(a_n) = (1, 2, 4, 11, ...)$.

(A joke: There is a documentary about Erdos that is titled, "N is a number". Of course N is a number...)

Some complexity theory notes:

"There exists a Hamiltonian cycle" is NP-complete.

"How many Hamiltonian cycles are there" is #P-complete.

"There exists graphs with n vertices satisfying some condition" is NEXP.

"The number of graphs with n vertices satisfying some condition" is #EXP

(NEXP $\neq$ EXP $\implies$ NP $\neq$ P.)

We have:

Theorem 169. *(Erdos) $Aut(G) = 1$ with high probability.*

Note, $a_n = \#$ unlabeled graphs on n vertices $= \sum_{\text{labeled } G \text{ with } n \text{ vertices}} \frac{1}{Aut(G)}$. But to compute for $Aut(G)$ will need to keep deciding whether two graphs G_1 and G_2 are isomorphic, which takes $e^{O(\sqrt{n})}$ time! (Recent: $e^{O(\log n)}$.)

Question: What is #EXP-complete?

Answer: Tilings. Consider a set of tiles T, and given a region Γ, we ask: Is Γ tilable with T?

Theorem 170. *Above tiling question is NP-complete.*

Remark 171. Well, If T is in the input, then it is NP-complete. If T is fixed however, and just Γ in the input, it is also NP-complete (feasible region by T?) Look at T to be the set with just 3×1 and 1×3 tiles.

Note however, that if T consists of just the 1×2 and 2×1 tiles, then it can be done in polynomial time. This is the perfect matching question.

Question: What is #P-complete?

Fix a set of tiles T, and take input Γ. Then:

Theorem 172. *The count how many ways to tile Γ with a fix set of tiles T is #P-complete.*

Theorem 173. *(Pak, Yang) For $\Gamma \subset \mathbb{Z}^3$, the number of domino tilings is #P-complete.*

Theorem 174. *(LMP) For a fixed set of tiles T, the rectangle $m \times n$ can be decided whether it is tilable with T or not in $\Theta(\log n + \log m)$ time.*

Theorem 175. *(Boas 1990) There exists set of tiles T such that # of tilings of an $m \times n$ rectangle is #EXP-complete.*

Theorem 176. *(Jed Yang) Whether there exists n, m such that the rectangle $m \times n$ is tilable with T or not is undecidable!*

WEEK 8 FRIDAY

53 Tilings.

Theorem 177. *(LMP) Let T be a fixed set of tiles. Then there exists a linear time algorithm to decide if $m \times n$ rectangle is tilable with T.*

Theorem 178. *(Yang) Given a set of tiles T whether there exists $m \times n$ rectangle is tilable with T is undecidable.*

Proof idea: Think of a Turing machine with an infinite tape. And imagine the lattice $\mathbb{Z}^2$ where the horizontal axis is the the space of the tape, and the vertical axis is time. So we can emulate Turing machines with tiles (automata with tiles). And as it turns out, the existence of a $m \times n$ such that it is tilable with T can be reduced to the halting problem, which is undecidable.

Theorem 179. *(Boas) There exists a fixed set of tiles T such that # of tilings of $m \times n$ is #EXP-complete.*

Proof idea: This can be turned into a 3-SAT problem.

How about proving the first theorem stated? Let us look at this curious theorem:

Theorem 180. *(Klarner's box theorem. deBruijn-K) Let $S \subset \{(m, n)\}$ be a set of ordered pairs (i.e. rectangles) that is closed under addition. Here addition means vertical and horizontal concatenation of rectangles, which can be done if the dimensions match up. Then there exists a finite set $F \subset S$ such that $S = \langle F \rangle$.*

This is like Hilbert basis theorem.

Example 181. Let $S = \{(m, n)$ tilable with $X\}$, where $X = \{(1, 4), (4, 1), (3, 3)\}$. Then take $F = \{(1, 4), (4, 1), (3, 3), (5, 5)\}$, we have $S = \langle F \rangle$, but $S \neq \langle X \rangle$ as some of the rectangles tilable from X cannot be obtained by addition in X.

Proof idea of the first theorem: First, another theorem:

Theorem 182. *(Barnes) Suppose $S = \langle F \rangle$, then there exists M, N, and $I \subset [0, M - 1]$, $J \subset [0, N - 1]$, such that for all $m \geqslant M, n \geqslant N$, the rectangle $m \times n$ is tilable with $F \iff m \mod M \in I$ and $n \mod N \in J$.*

Now, if we can precompute M, N, I, J then we would have a linear time algorithm. Done.

However, we cannot really precompute this: Let $T = \{t_1, \dots, t_l\}$ tiles, and $k = \sum |t_i|$, and let $\alpha(T) = \begin{cases} \min\{mn : m \times n \text{ is tilable with } T\} \\ 0 \text{ otherwise} \end{cases}$. How big is this $\alpha(T)$? Well, $\alpha > 2^{2^{2^{\cdot^{\cdot^{2}}}}}$ (Yang). So from a computational point of view, "there is no precomputing M, N", as they are large.

Corollary 183. *(From Yang's theorem above) There exists a set of tiles T such that the existence of $m \times n$ tilable with T is independent of ZFC/PA.*

WEEK 9 MONDAY

54 Computational aspects of balanced words in a group.

Let G be an infinite group with finite set S that generates G. Denote $a_n(G, S)$ to be the number of words in S of length n that equals 1 in G. Certainly we have $0 \leqslant a_n \leqslant |S|^n$. Recall we showed

Theorem 184. *[GP] There exists $G \subset SL(4, \mathbb{Z})$ and finite generating set $S \subset G$ such that $\{a_n(G, S)\}$ is not P-recursive.*

What about the computational aspects of a_n?

Theorem 185. *(Mihalkova) "For (G, S) as above such that $a_n = 0$ for all $n \geqslant 1$" is undecidable.*

Or: "There exists (G, S) such that $a_n = 0$ for all $n \geqslant 1$" is independent of ZFC.

Problem 1: Given (G, S), decide whether $\{a_n(G, S)\}$ is P-recursive (or ADE, $\mathbb{N}$-rational).

Conjecture 186. *Problem 1 is undecidable.*

Problem 2: For (G, S) as above. Given n, compute $a_n(G, S)$. (Well, we could just consider all $|S|^n$ possible words)

Conjecture 187. *Problem 2 is #EXP-complete. (Well it is in #EXP, but why complete?)*

Theorem 188. *If EXP $\neq \oplus$EXP, then $\{a_n\}$ cannot be computed in poly(n) time.*

Remark 189. Here $\oplus$ means parity. Note P $\subset \oplus$P $\subset$ #P.

Example 190. Counting # of Hamiltonian cycles (HC) in graph Γ is #P-complete.

Example 191. $\oplus$HC in graph Γ is in $\oplus$P. ($\oplus$HC means whether the number of HC is even or odd.)

Example 192. # 3-colorings is #P-complete. However $\oplus$ 3-colorings is in P. This is because # 3-colorings = 6 # exactly 3-colorings + # 2-colorings, and that $\oplus$2-colorings is in P.

Example 193. # Perfect matching in bipartite Γ is #P-complete; $\exists$PM in Γ is in P; and $\oplus$PM in Γ is also in P. This is because #PM = Permanent(M_Γ) = Determinant(M_Γ) mod 2. This is by accident.

Theorem 194. *(#PM mod 3) is $\oplus$P-hard.*

Something similar to $\oplus$ problems is the uniqueness problems.

Example 195. Both UHC (there exists unique Hamiltonian cycle) and U3C (there exists unique 3-coloring) are both NP-hard

Theorem. *(TODA) P = $\oplus$P $\implies$ PH = BPP ($\approx$P). Here PH = polynomial hierarchy, and BPP = bounded probability polynomial.*

Proof idea: By using oracles.

Corollary 196. *There exists (G, S) as above such that $\{a_n(G, S)\}$ is not ADE, unless EXP = $\oplus$EXP.*

Remark 197. This is fascinating in combinatorics, as complexity is a notion not really present before.

As it turns out,

Theorem 198. *(GP) There exists (G, S) such that a_n is not ADE. (Without the complexity condition above.)*

Conjecture 199. *$\oplus$Problem 2 is $\oplus$EXP-complete.*

Conjecture 200. *Suppose $S \subset \mathbb{R}^2$ a finite set of points, then # triangulations on S is #P-complete.*

However, this is sensitive to what S is: If S is convex, then this is just the $|S|$-th Catalan number.

Conjecture 201. *Let $S_n = n \times n$ grid. Then # triangulations of S_n is #EXP-complete.*

Whether the sequence $\{\#trian.(S_n)\}$ is P-recursive or not, this is unknown.

However, we do have:

Theorem 202. *For $S_{n,k} = k \times n$ grid, with k fixed, then #triang.$(S_{n,k})$ can be computed in polynomial time.*

WEEK 9 WEDNESDAY

55 What is "is"?

Let $\{a_n\}$ be an integer sequence, recall we have been asking the kind of question: Is $\{a_n\}$ P-recursive, is it ADE? But what do we mean by that?

Aside, which languages have the word "is"?

Have "is": En, Fr, Vn, Sp, Ge, Ch.
Do not have "is": Ru, He, Kz, Ar, Tr.

Which of the following sequences are P-recursive?

1, 2, 3, 4, 5, ... $a_n = n$
1, 2, 4, 8, 16, ... $a_n = 2^n$
1, 2, 5, 14, 42, ... $a_n = \frac{1}{n+1}\binom{2n}{n}$
1, 2, 6, 24, 120, ... $a_n = n!$
2, 3, 5, 7, 11, ... primes
Fermat primes

Well, the ones are that are obvious are already given as P-recursive!

Note $\{a_n\}$ is given as

(1) Formula
(2) Combinatorial interpretations
(3) Logical formula
(4) ...more...

If it is already given as a formula (1), then it is not hard to see if it is P-recursive. However, for (2)-(4) it may not be as clear.

Conjecture 203. *The primes $\{p_n\}$ is not ADE. (It is not P-recursive however.)*

We believe in this conjecture as primes should not come from something nice, but we cannot prove it.

Problem 1. Suppose we know a graph Γ has a Hamiltonian cycle already. Find such HC. This is not easy.

Problem 2. Given a HC in Γ, decide whether Γ has another HC. Still hard!

56 Knowledge vs. Certificate.

Given $A = \sum a_n t^n$ satisfying $c_0 A + c_1 A' + \cdots + c_r A^{(r)} = d$. Is A algebraic? Or, given A is algebraic satisfying $c_0 A^r + c_1 A^{r-1} + \cdots + c_r A + c_0 = 0$, is A rational? Well we do not have sufficient information yet.

We look at a series of examples.

Example 204. Non-crossing matchings on $2n$ points in a line. There are Catalan(n) of these.

Example 205. Non-nesting matchings on $2n$ points in a line. There are also Catalan(n) of these.

Example 206. Connected matchings on $2n$ points in a line (where each point can go to any other point through the arcs). Denote these $Con(n)$. We have

n	1	2	3	4
$Con(n)$	1	4	27	248

Example 207. Crossing matchings on $2n$ points in a line (where every arc crosses at least one other arc). Denote these as $Cro(n)$. We have

n	1	2	3	4
$Cro(n)$	1	4	31	288

Theorem 208. *(Klazar) $\{Con(n)\}$ and $\{Cro(n)\}$ are not P-recursive.*

Theorem 209. *(Klazar) $\{Con(n)\}$ and $\{Cro(n)\}$ are ADE.*

Proof idea of these. First note for $E(t) = \sum Cro(n)t^n$ and $F(t) = \sum Con(n)t^n$, we have $E' = \frac{E^2+2E-t}{2tE}$ and $F' = \frac{-t^2F^3+F-1}{2t^3F^2+2t^2F}$. Firstly in this form, it shows that E and F are ADE. But also when it is in this form, that $E' = \frac{*}{*}$ and $F' = \frac{*}{*}$ are nontrivial rational functions, which means $E^{(r)}$ and $F^{(r)}$ becomes more complicated. And thus E and F cannot be P-recursive.

This gives a strategy: If we can write $A' = \frac{poly(A,t)}{poly(A,t)}$, then we may be able to decide A is P-recursive or not.

Example 210. Simple permutations. A permutation τ has a block if it contains a section of consecutive integers. For instance $\tau = (2\,\underline{6\,4\,7\,5}\,1\,3)$ contains a block as indicated. We say τ is simple if it contains no blocks. Let a_n = number of simple permutations in $\mathfrak{S}_n$. We have $a_n \sim \frac{n!}{e^2}(1+o(1))$. For $A = \sum a_n t^n/n!$, if we modify a_0 and a_1 to something else, we can get $A' = \frac{A^2}{t-(1+t)A}$. So we see that A is ADE, and by our strategy above, A is not P-recursive.

(Friday is Thanksgiving holiday.)

WEEK 10 MONDAY

57 On classifications of generating functions and modularity conditions.

Question. Suppose we known $\{a_n\}$ is of class (i), and we some modularity conditions of $\{a_n\}$. Then do we know whether $\{a_n\}$ is in a smaller class (i-1)?

Example 211. $\{a_n\}$ is such that $\{a_n \mod p\}$ is periodic for all prime $p \not\Rightarrow \{a_n\}$ periodic.

Indeed. Take $a_n = n$.

Okay, but $a_n = n$ is P-recursive. So then, is there a non P-recursive example?

Well for $a_n = 2^{2^n}$, here a_n mod prime is periodic for all prime (Fermat little theorem). But a_n is not P-recursive and not periodic (grows too fast!)

(Or take something such that under mod prime is 1 1 1 1 1 1.. by using Chinese remainder theorem. Say $a_k \equiv 1$ mod the first k primes, and make a_k grow large enough.)

Example 212. $\{a_n\}$ ADE and $a_{cn+d} = 0 \mod p$ for infinitely many $(c, d, p) \not\Rightarrow \{a_n\}$ P-recursive.

Take $a_n = p(n)$, number of partitions of n, which is not P-recursive. But $p(5n + 4) \equiv 0 \mod 5$, $p(7n+6) \equiv 0 \mod 7$,..., and there are infinitely many of these.

Example 213. Eventual periodic in mod prime $\not\Rightarrow$ periodic.

Indeed, take $a_n = n!$

Example 214. $\{a_n\}$ P-recursive, eventual periodic mod p, for all prime $p \not\Rightarrow$ rational.

Example 215. $\{a_n\}$ algebraic, eventual periodic mod p, for all prime $p \implies$ rational.

Indeed, this is actually a theorem in algebraic geometry (and not by elementary combinatorics.) (Deligne?)

Example 216. P-recursive vs P-recursive mod p prime (over various fields)

$\Longleftarrow$ is true.

Example 217. Recall $a_n = Con(n)$. We have:

Theorem 218. *$\{a_n \mod 2^k\}$ is P-recursive for each $k \geqslant 1$.*

But still, $\{a_n\}$ itself is not P-recursive.

Example 219. Somos numbers.

Let $a_0 = a_1 = a_2 = a_3 = 1$, and $a_n = \frac{a_{n-3}a_{n-1}+a_{n-2}^2}{a_{n-4}}$. These are called the Somos-4 numbers. It goes like this: 1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, ... It is not easy to see that they are all integers! Here $a_n \sim e^{\Theta(n^2)}$. Well, $\{a_n\}$ is ADE. But because a_n grows too fast, it cannot be P-recursive. We also have the following:

Theorem 220. *(1) $a_n \in \mathbb{N}$.*
(2) a_n = # perfect matchings in Γ_n (some variation of the Aztec diamond.)
(3) [Robinson] $\{a_n \mod m\}$ is periodic for all $m \in \mathbb{N}$.

So we conclude with the following moral: Modularity condition tells us nothing.

(Also, Somos is a person who really liked sequences.)

WEEK 10 WEDNESDAY

58 Relation to binomial identities.

Consider $\underbrace{\sum_{k=0}^{n}\binom{n}{k}}_{a_n} = 2^n$. Question: What class is $\{a_n\}$? From the LHS, it is a binomial sum, and hence diagonal. But from the RHS, it is clear that is is rational!

How about $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$? From the LHS, it is again a binomial sum. But the RHS shows it is algebraic that we showed before.

How about $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = \binom{3n}{n,n,n}(-1)^n$ (Dixon's identity)? The LHS says it is a binomial sum, and the RHS shows it is P-recursive.

Let us define $S_r(n) = (-1)^n\sum_{k=0}^n(-1)^k\binom{2n}{k}^r$. Note $S_1(n) = 0$. Here $S_r(n)$ generalizes the LHS of Dixon's identity.

Theorem 221. *(deBruijn) For all $r \geqslant 4$, $S_r(n) \neq \prod_{i=1}^p(\alpha_i n + \beta_i)!^{c_i}$ for any $\alpha_i, \beta_i \in \mathbb{N}$, any $c_i \in \mathbb{Z}$, and any p.*

So for higher r, $S_r(n)$ is "nontrivial" But how did deBruijn prove this?

Note the real theorem is actually: For each $r \geqslant 1$, $S_r(n) \sim 2(\cos\frac{\pi}{2r})^{2nr+r-1}2^{2-r}(\pi n)^{\frac{1-r}{2}}\frac{1}{\sqrt{r}}$ (?) as $n \to \infty$. This is a result in analysis.

Compare with Stirling's approximation, $n! \sim \sqrt{\pi n}n^n e^{-n}$. So by plugging this approximation, we need that $S_r(n) \sim \lambda^n n^\alpha(\log n)^\beta$ for $\lambda \in \mathbb{Q}$. And note $S_r(n) \sim ((\cos(\pi/2r))^{2r})^n\cdots$, so for $r = 4$ we have $\cos(\pi/8)^8 \notin \mathbb{Q}$. So we can conclude deBruijn's theorem.

Theorem 222. *(Dixon's identity) $\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = \binom{3n}{n,n,n}(-1)^n$.*

How do we prove this? This is in fact a consequence of the following:

Lemma 223. *(MacMahon's Master Theorem, MMT) Let $B = (b_{ij}) \in M_{m\times m}(\mathbb{C})$, and $x_1, \ldots, x_m$ variables. Take $G(k_1, \ldots, k_m) = [x_1^{k_1}\cdots x_m^{k_m}]\prod_{i=1}^m(b_{i1}x_1 + \cdots + b_{im}x_m)^{k_i}$. Then*

$$\sum_{(k_1,\ldots,k_m)} G(k_1, \ldots, k_m)t_1^{k_1}\cdots t_m^{k_m} = \frac{1}{\det(I - TB)},$$

where $T = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{pmatrix}$.

Now we show MMT implies Dixon's identity:

Take $B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & - & 1 \\ 1 & -1 & - \end{pmatrix}$. Then $G(2n,2n,2n) = [x_1^{2n}x_2^{2n}x_3^{2n}]\prod(x_2-x_3)^{2n}(x_3-x_1)^{2n}(x_1-x_2)^{2n}$. By looking at what contributes to the term $x_1^{2n}x_2^{2n}x_3^{2n}$ from the product, we see that for each k, we will get $(x_2^k x_3^{2n-k}) \cdot (x_1^{2n-k}x_3^k) \cdot (x_1^k x_2^{2n-k})$, and get $(-1)^k\binom{2n}{k}$ from each of the three terms. So by summing over k, we get $G(2n,2n,2n) = \sum_k (-1)^{3k}\binom{2n}{k}^3$, which is LHS of Dixon's.

Now for the RHS, note that $\det(I-TB) = \det\begin{pmatrix} 1 & -t_1 & t_1 \\ t_2 & 1 & t_2 \\ -t_3 & t_3 & 1 \end{pmatrix} = 1 + (t_1t_2 + t_2t_3 + t_1t_3)$. Hence $\frac{1}{\det(I-TB)} = \sum(-1)^n(t_1t_2+t_2t_3+t_1t_3)^n$, which we see that

$$[t_1^{2n}t_2^{2n}t_3^{2n}]\frac{1}{\det(I-TB)} = \binom{3n}{n,n,n}(-1)^n.$$

Hence we have Dixon's identity.

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59 Proof of MacMahon's master theorem.

Recall MacMahon's master theorem from last time:

Lemma. *Let $B = (b_{ij}) \in M_{m\times m}(\mathbb{C})$, and $x_1,\dots,x_m$ variables. Take $G(k_1,\dots,k_m) = [x_1^{k_1}\cdots x_m^{k_m}]\prod_{i=1}^m(b_{i1}x_1+\cdots+b_{im}x_m)^{k_i}$. Then*

$$\sum_{(k_1,\dots,k_m)} G(k_1,\dots,k_m)t_1^{k_1}\cdots t_m^{k_m} = \frac{1}{\det(I-TB)}, \text{ where } T = \begin{pmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{pmatrix}.$$

And it was used to prove Dixon's identity. But why is MMT true? Let us meditate on this:

Example 224. Suppose $B = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix}$ is a diagonal matrix. Then $G(k_1,\dots,k_m) = \lambda_1^{k_1}\cdots\lambda_m^{k_m}$, so

$$\sum_{(k_1,\dots,k_m)} G(k_1,\dots,k_m)t_1^{k_1}\cdots t_m^{k_m} = \frac{1}{1-\lambda_1t_1}\cdots\frac{1}{1-\lambda_mt_m}.$$

But note $\det(I-TB) = \det\begin{pmatrix} 1-\lambda_1t_1 & & \\ & \ddots & \\ & & 1-\lambda_mt_m \end{pmatrix} = (1-\lambda_1t_1)\cdots(1-\lambda_mt_m)$.

Hence MMT is true for diagonal matrices B.

Now, we have $\sum_{l\geqslant 0} tr(S^lB) = \frac{1}{\det(I-B)}$, where S^lB is a symmetric power of B, and this is a continuous result (treating b_{ij} as variables). So suffices to prove MMT for diagonalizable matrices, as they are dense. But above is also stable under $GL(m,\mathbb{C})$ via conjugation. So we just need to show MMT is true for diagonal cases, which we did!

60 WZ algorithm and the WZ pair.

In general, how can we show the equality: $\sum_k f(n,k) = r(n)$? Or, by setting $F(n,k) = \frac{f(n,k)}{r(n)}$, the equality $\sum_k F(n,k) = 1$?

Wilf and Zeilberger's idea: Suppose $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$ for all k for some G such that $\lim_{k\to\pm\infty} G = 0$, then

Theorem 225. *(With F hypergeometric) This works $\iff G(n,k) = F(n,k)R(n,k)$ where $R(n,k)$ is rational.*

F and G is called a WZ pair.

Example 226. Showing $\sum_k \binom{n}{k}^2 = \binom{2n}{n}$, or showing

$$\sum_k \frac{(n+b)!(n+c)!(b+c)!(-1)^k}{(n+k)!(n-k)!(b+k)!(b-k)!(c+k)!(c-k)!} = \binom{n+b+c}{n,b,c}.$$

(Zeilberger has a machine that does this: Shalosh Ben Ekhad.)

61 Summary and open questions.

We looked at various classes of generating functions, and analyzed them by looking at their asymptotics, etc. We also looked at how they relate to combinatorial problems like balanced words, tilings, walks on graphs, and automata.

However:

Thought (1). We still do not know the asymptotics of general P-recursive cases: We think:

Conjecture 227. *If (a_n) is P-recursive, then $a_n \propto (n!)^s\lambda^n\mu^{Q(n^{1/m})}n^\alpha(\log n)^\beta C$, where C is a period, $\lambda,\mu\in\overline{\mathbb{Q}}$, $Q\in\mathbb{Z}[t]$, $\alpha\in\mathbb{Q}$, $\beta\in\mathbb{N}$.*

Note we often can use asymptotics to show whether a sequence is NOT of a class. But to show membership we need a structural approach.

Thought (2). Other properties. Recall

- Connected matchings: We showed this is ADE but not P-recursive by expressing its derivative in a specific form.

- # walks: We analyzed it by taking mod m.

But these techniques are quite ad hoc, and not general methods.

Thought (3). For P-recursive, we know almost. But for ADE, we know almost nothing. Are primes ADE? (Or other questions such as: Are there infinitely primes of n^2+1? Infinitely many Fibonacci primes? Fermat prime?) Recall q_n = # binary partitions of n. Is (q_n) ADE? We do not know. We have q_n satisfying some functional equation, however: $q_{2n} = q_{2n-1} + q_n$ and $q_{2n+1} = q_{2n}$.

(End)