The Robinson–Schensted–Knuth correspondence

* We have seen that the plactic classes are indexed by a tableau
$$\pi : A^* \to Pl(A) = A^*/\equiv \quad , \quad \pi^{-1}(t) \subseteq A^*$$
* next we show that words in $\pi^{-1}(t)$ are also indexed by a tableau.

A tableau is <u>standard</u> if its entries are $1, 2, \dots, n$ each appearing once.
Let $Tab(\lambda, A)$, $(STab(\lambda))$ be the set of tableau, (standard) tableau shape λ entries in A $\{1, 2, \dots, n\}$

<u>recording tableau:</u>

At each step of Schensted's insertion we record where the entry is added in the shape λ. The outcome is a standard tableau $Q(w)$ of shape λ

ex 13214 $P(w)$: $1 \to 13 \to \begin{matrix}12\\3\end{matrix} \to \begin{matrix}11\\2\\3\end{matrix} \to \begin{matrix}114\\2\\3\end{matrix}$

$Q(w)$: $1 \quad 12 \quad \begin{matrix}12\\3\end{matrix} \quad \begin{matrix}12\\3\\4\end{matrix} \quad \begin{matrix}125\\3\\4\end{matrix}$

So we have a map $\rho : A^* \to \bigsqcup_\lambda Tab(\lambda, A) \times STab(\lambda)$
$$w \mapsto (P(w), Q(w))$$

Theorem 4 ρ is a bijection called the Robinson–Schensted correspondence.

pf we describe ρ^{-1} by example

$\begin{matrix}123\\13\\3\end{matrix} \quad \begin{matrix}125\\36\\4\end{matrix} \to \begin{matrix}123 \leftarrow 3\\1\\3\end{matrix} \quad \begin{matrix}123^{\leftarrow 3}\\1\\3\end{matrix} \quad \begin{matrix}125\\3\\4\end{matrix} \to \begin{matrix}12\\1 \leftarrow 3\end{matrix} \quad \begin{matrix}12\\3\\4\end{matrix}, \quad \begin{matrix}12\\3 \leftarrow 1\end{matrix}, \quad \begin{matrix}12^{\leftarrow 1}\\3\end{matrix}$

$12 \leftarrow 3 \quad \begin{matrix}12\\3\end{matrix}, \quad 13^{\leftarrow 2} \quad 3 \quad 33 \quad 133$

$21 33 \to \boxed{13 2 1 33}$

idea given row v & letter y, there exists unique row v' & letter x such that $yv \equiv v'x$.

Cor 1 Q induces a bijection between the plactic class $\pi^{-1}(t)$ and $STab(\lambda)$ where λ is the shape of t.

Let $f_\lambda := |STab(\lambda)|$

* If we restrict ρ to the standard words of $\{1,2,\dots,n\}$ (i.e. permutations $\mathfrak{S}_n$ of n) then we get the bijection

$$\mathfrak{S}_n \longleftrightarrow \bigsqcup_\lambda STab(\lambda) \times STab(\lambda)$$

which implies $$n! = \sum_{\lambda \text{ partition of } n} (f_\lambda)^2$$ (Identity of Fröbenius)

$123 \to (123, 123)$, $213 \to \left(\begin{smallmatrix}1&3\\2\end{smallmatrix}, \begin{smallmatrix}1&3\\2\end{smallmatrix}\right)$, $132 \to \left(\begin{smallmatrix}1&2\\3\end{smallmatrix}, \begin{smallmatrix}1&2\\3\end{smallmatrix}\right)$,

$231 \to \left(\begin{smallmatrix}1&3\\2\end{smallmatrix}, \begin{smallmatrix}1&2\\3\end{smallmatrix}\right)$, $312 \to \left(\begin{smallmatrix}1&2\\3\end{smallmatrix}, \begin{smallmatrix}1&3\\2\end{smallmatrix}\right)$, $321 \to \left(\begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\\3\end{smallmatrix}\right)$

Note that if $\sigma = 231$, $\sigma^{-1} = 312$ & $P(231) = Q(312)$, $Q(231) = Q(312)$.

This symmetry holds in general:

Theorem 5 For $\sigma \in S_n$, $Q(\sigma) = P(\sigma^{-1})$.

To sketch the proof we use Greene's thm.

- view σ in two line notation $\begin{pmatrix}1&2&\cdots&n\\ \sigma_1&\sigma_2&\cdots&\sigma_n\end{pmatrix}$ or $\begin{pmatrix}\sigma^{-1}\\ id\end{pmatrix}$

  (biword lex ordered wrt the top) (biword lex ordered wrt to bottom.)

- Given $\begin{pmatrix}u\\v\end{pmatrix}$ $u, v \in A^*$

  let $\begin{pmatrix}u'\\v'\end{pmatrix}$ lex rearrangement w.r.t top

  $\begin{pmatrix}u''\\v''\end{pmatrix}$ " " " bottom

  ex $\begin{pmatrix}2&1&1&3\\1&3&6&4\end{pmatrix}$ $\begin{pmatrix}u'\\v'\end{pmatrix} = \begin{pmatrix}1&1&2&3\\3&6&1&4\end{pmatrix}$, $\begin{pmatrix}u''\\v''\end{pmatrix} = \begin{pmatrix}2&1&3&1\\1&3&4&6\end{pmatrix}$

Lemma Given $\begin{pmatrix}u\\v\end{pmatrix}$ then $P(v')$ and $P(u'')$ have the same shape.

Pf

we show $\ell_k(v') = \ell_k(u'')$ for all k.

Let $\beta = v_{i_1} \cdots v_{i_r}$ be $\leq$ subword of v', then $\alpha = u_{i_1}, \dots, u_{i_r}$ is also $\leq$

so $u_{i_1} \leq u_{i_2} \cdots \leq u_{i_r}$, $v_{i_1} \leq v_{i_2} \quad \leq v_{i_r}$, so α is also a $\leq$ subword of u''.

— we have a correspondence $\leq$ subwords of v' and u'' respectively.

$\Rightarrow \ell_k(v') = \ell_k(u'')$ for all k.

<u>pf</u> Thm 5 for perm $\sigma \in S_n$ $\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} id \\ \sigma \end{bmatrix}$ & $\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \begin{bmatrix} \sigma^{-1} \\ id \end{bmatrix}$

& $\begin{bmatrix} u'(k) \\ v'(k) \end{bmatrix} = \begin{bmatrix} 1 2 \cdots k \\ \sigma_1 \cdots \sigma_k \end{bmatrix}$, $\begin{bmatrix} u''(k) \\ v''(k) \end{bmatrix} = \begin{bmatrix} \sigma^{-1}|_{[1,k]} \\ (\sigma_1 \cdots \sigma_k)_{\le} \end{bmatrix}$ by Lemma $P(\sigma_1 \cdots \sigma_k)$ and $P(\sigma^{-1}|_{[1,k]})$ have same shape

but $P(\sigma^{-1}|_{[1,k]})$ and $P(\sigma^{-1}|_{[1,k+1]})$ differ by adding $k+1$

so at the end $P(\sigma^{-1}) = Q(\sigma)$. ☒

ex $\sigma = 31452$

	3	31	314	3145	31452
P	3	1 3	14 3	145 3	125 34

$\sigma^{-1} = 25134$

1	21	213	2134	25134
1	1 2	13 2	134 2	134 25

We can generalize RSK to biwords $\binom{u}{v}$ where u' is not standard but $\le$

ex. $\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 3 & 2 & 3 & 1 & 2 & 2 & 2 \end{pmatrix}$ ← build Q, ← build P

$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ ↗ entries in $\mathbb{N}$

1	11	113	112 3	1123 3	1113 2 3	1112 23 3	111222 23 3
1	11	111	111 2	1112 2	1112 2 3	1112 23 3	111233 23 3

<u>Thm 6</u> There is a bijection between $\mathbb{N}$-matrices $A = (a_{ij})$ and pairs (P, Q) in $STab(A) \times STab(A)$ such that

j occurs in P $\sum_i a_{ij}$ times

i " " Q $\sum_j a_{ij}$ times

<u>Schur functions</u> Let $x_1, x_2, \ldots$ be commuting variables, for $w \in A^*$ let $\underline{w}$ be the commutative image of w, $a_i \mapsto x_i$
($a_i \in A$)

<u>Def</u> (Schur function) let $s_\lambda(x_1, x_2, \ldots, x_n) = \sum_{t \in STab(\lambda)} \underline{t}$ for a partition λ of m

ex $s_{21}(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + 2x_1x_2x_3 + x_2^2 x_1 + x_3^2 x_1 + x_2^2 x_3$

11 2	11 3	12 3	13 2	12 2	13 3	22 3

Prop $S_\lambda(x_1,\dots,x_n)$ is a symmetric polynomial in $x_1,\dots,x_n$

Corollary of Thm. 6

$$\prod_{i,j}(1-x_iy_j)^{-1} = \sum_{\substack{\text{partition}\\ \lambda}} S_\lambda(x_1,\dots,x_n)\, S_\lambda(y_1,\dots,y_n)$$

(Cauchy identity)

- $\prod_{i,j}(1-x_iy_j)^{-1}$: generating function $\mathbb{N}$-matrices, weight $a_{ij} \to x_iy_j$
- $S_\lambda(x_1,\dots,x_n)$: generating function tableaux P
- $S_\lambda(y_1,\dots,y_n)$: generating function tableaux Q

* From representation theory one can show that $S_\lambda(x_1,\dots,x_n)\, S_\mu(x_1,x_2,\dots,x_n)$ expand positively in the Schur functions.

$$S_\lambda(x_1,x_2,\dots,x_n)\, S_\mu(x_1,x_2,\dots,x_n) = \sum_\nu c_{\lambda\mu}^{\nu}\, S_\nu(x_1,x_2,\dots,x_n)$$

where $c_{\lambda\mu}^{\nu} \in \mathbb{N}$ Do they count something?

* the coefficients $c_{\lambda\mu}^{\nu}$ are of great interest, called the Littlewood-Richardson coefficients

Littlewood, Richardson 1934 gave a combinatorial rule to compute them w/o proof.

- A complete correct proof given by M.P. Schutzenberger, 1977 using $Pl(A)$
- Saturation conjecture $c_{\lambda\mu}^{\nu} \neq 0 \iff c_{N\cdot\lambda\, N\cdot\mu}^{N\cdot\nu} \neq 0$ (Knutson-Tao)

Littlewood-Richardson rule

Def. let $S_\lambda(A) = \sum_{t\in Tab(\lambda,A)} t$ in $\mathbb{Z}[Pl(A)]$ (plactic Schur function)

$S_\lambda(A)$ is a projection in $\mathbb{Z}[Pl(A)]$ of

$$\underline{S}_t(A) = \sum_{Q(w)=t} w \quad \text{in } \mathbb{Z}\langle A\rangle$$

(free Schur functions)

Main Theorem 7 The plactic Schur functions span a commutative subalgebra of $\mathbb{Z}[Pl(A)]$ and

$$S_\lambda(A)\, S_\mu(A) = \sum_\nu c_{\lambda\mu}^{\nu}\, S_\nu(A)$$

i.e. $c_{\lambda\mu}^{\nu} = \{\text{ factorisations } t't'' = t \mid t' \in T_\lambda(A),\ t'' \in T_\mu(A),\ \text{fixed } t \in T_\nu(A)\}$

ex $A = \{1<2<3\}$

$$S_{21}(A)\, S_{21}(A) = S_{2211} + S_{222} + S_{3111} + S_{33} + S_{411} + S_{42} + 2\, S_{321}$$

why $c^{321}_{21,21} = 2$ $\quad t = \begin{matrix} 1\,1\,1 \\ 2\,2 \\ 3 \end{matrix} \in STab(321, A)$

$$\begin{matrix} 1\,1\,1 \\ 2\,2 \\ 3 \end{matrix} = \begin{matrix} & & 1\,1 \\ & 2 & \\ 1\,2 & & \\ 3 & & \end{matrix} \quad , \quad \begin{matrix} 1\,1\,1 \\ 2\,2 \\ 3 \end{matrix} = \begin{matrix} & 1\,1 \\ & 2 \\ 1\,3 & \\ 2 & \end{matrix}$$

* The proof of Theorem ∓ uses the following Lemma:

let $u \sqcup\!\sqcup v$ denote the "shuffle" of words u & v

$$ab \sqcup\!\sqcup cde = abcde + acbde + cadbe + cdaeb + cdeab$$

then

<u>Lemma</u> A', A'' two subalphabets s.t. $a' < a''$ for $a' \in A'$, $a'' \in A''$, then for $t' \in Tab(A')$, $t'' \in Tab(A'')$ we have

$$\Big(\sum_{P(w')=t'} w'\Big) \sqcup\!\sqcup \Big(\sum_{P(w'')=t''} w''\Big) = \sum_{t \in Sh(t',t'')} \sum_{P(w)=t} w$$

$Sh(t',t'')$ is the set of tableau t with $t|_{A'} = t'$, $P(t|_{A''}) = t''$.

(shuffle plactic class of A' with plactic class of A'' gives union of plactic classes of A)