The Plactic Monoid (M.P. Schützenberger)

A totally ordered alphabet $\{a_1 < a_2 < \cdots < a_n\}$, $\{1 < 2 < \cdots < n\}$

Motivation Given a $w \in A^*$, what is the length of the longest nondecreasing subword of w?
(complexity of finding such a subword is $O(n \ln n)$)

132541, longest nondecreasing subword has length 3.

C. Schensted (now EaEa) gave an algorithm that finds the length without finding the sequence using Young tableau

Definition (tableau)

* a nondecreasing word $u \in A^*$ is called a row
* rows $u = x_1 \cdots x_r$, $v = y_1 \cdots y_s$, $u \triangleleft v$ if $r \geq s$, $x_i < y_i$

$$\begin{array}{l} x_1 x_2 \cdots \overset{\leq}{\cdots} x_r \\ y_1 y_2 \cdots y_s \end{array} \quad \wedge$$

* every w has a unique factorisation $w = u_1 u_2 \cdots u_k$ into rows u_i of maximal length)
  a tableau is a word w such that $u_1 \triangleright u_2 \triangleright \cdots \triangleright u_k$

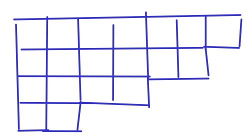
ex $68\ 4556\ 223357\ 1112444 \mapsto$

$$\begin{array}{lllllll} 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 5 & 7 \\ 4 & 5 & 5 & 6 \\ 6 & 8 \end{array}$$

tableau has shape 7, 6, 4, 2 (length of rows)

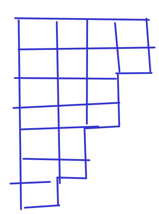
columns are increasing subwords

* A Young tableau is a tableau with partition shape λ



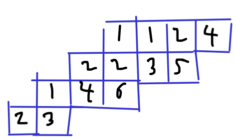
λ

* λ' denotes the conjugate partition of λ ($\lambda' = 4433221$)



λ'

* A skew Young tableau is a tableau of skew shape λ/μ ($\mu \leq \lambda$) ex $\lambda = 7642$, $\mu = 321$



Schensted's insertion algorithm

input $w \in A^*$ output a tableau $t = P(w)$

(length of first row = length of longest nondecreasing subword of w)

elementary step:

- $v = y_1 y_2 \cdots y_s$ row, insert x

$$y_1 y_2 \cdots y_s \leftarrow x$$

$$P(v) = \begin{cases} y_1 y_2 \cdots y_s\, x & \text{if } vx \text{ is a row } (y_s \le x) \\ y_1 y_2 \cdots x\, y_{i+1} \cdots y_s \leftarrow y_i & \text{if } y_i \text{ leftmost letter of } v,\ y_i > x \end{cases}$$

"x bumps y_i"

- insert x in a tableau: insert x in first row, if y_i is bumped, insert y_i second row ----

ex insert $x=3$ in

$$\begin{array}{llllllll} 1 & 1 & 1 & 2 & 4 & 4 & 4 & \leftarrow 3 \\ 2 & 2 & 3 & 3 & 5 & 7 \\ 4 & 5 & 5 & 6 \\ 6 & 8 \end{array}$$

$$\begin{array}{lllllll} 1 & 1 & 1 & 2 & 4 & 4 & 4 \\ 2 & 2 & 3 & 3 & 5 & 7 \\ 4 & 5 & 5 & 6 \\ 6 & 8 \end{array} \leftarrow 3 \qquad \begin{array}{llllll} 1 & 1 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 3 & 3 & 5 & 7 \\ 4 & 5 & 5 & 6 \\ 6 & 8 \end{array} \leftarrow 4 \qquad \begin{array}{lllllll} 1 & 1 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 3 & 3 & 4 & 7 \\ 4 & 5 & 5 & 6 \\ 6 & 8 \end{array} \leftarrow 5$$

$$\begin{array}{lllllll} 1 & 1 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 3 & 3 & 5 & 7 \\ 4 & 5 & 5 & 5 \\ 6 & 8 \end{array} \leftarrow 6 \qquad \begin{array}{lllllll} 1 & 1 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 3 & 3 & 5 & 7 \\ 4 & 5 & 5 & 5 \\ 6 & 6 \\ 8 \end{array}$$

General case: given $w = x_1 \cdots x_n$

insert x_1 to $\emptyset$ get $P(x_1)$

insert x_2 to $P(x_1)$ --- $P(x_1 x_2)$

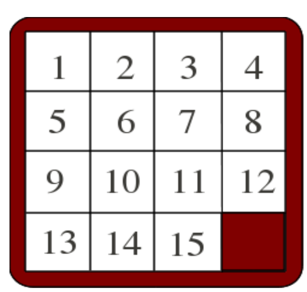
⋮

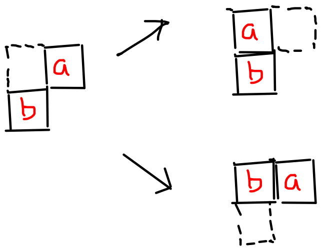
insert x_n to $P(x_1 \cdots x_{n-1})$ get $P(x_1 x_2 \cdots x_n)$

example $P(132541)$

$$1 \qquad 1\,3 \qquad \begin{array}{ll} 1 & 2 \\ 3 \end{array} \qquad \begin{array}{lll} 1 & 2 & 5 \\ 3 \end{array} \qquad \begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 \end{array} \qquad \begin{array}{lll} 1 & 1 & 4 \\ 2 & 5 \\ 3 \end{array}$$

Schensted's insertion algorithm (jeu-de-taquin version)



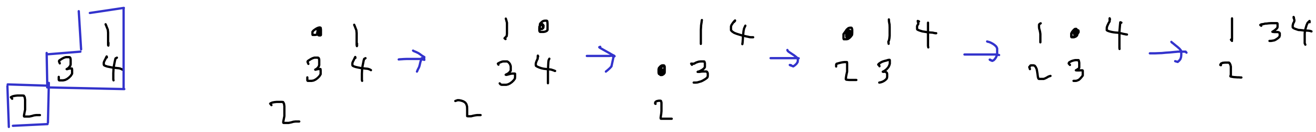


if $a \le b$

if $a > b$

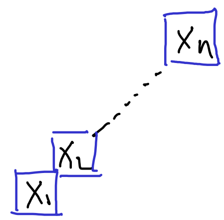
(do a move to preserve a tableau)

"jeu de taquin" is a set of rules to transform a skew tableau into a Young tableau

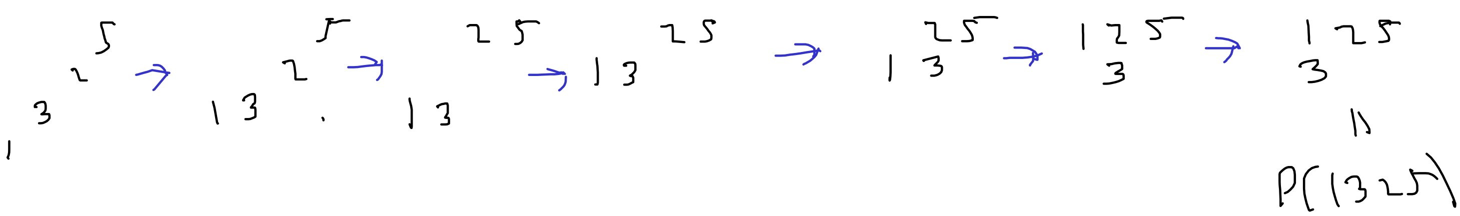


ex jeu de taquin models a "corporate restructuring"

Claim Schensted insertion $\equiv$ jeu de taquin
$w = x_1 x_2 \cdots x_n$



ex 1325



Thm1 - The maximal length of a nondecreasing subword of w is the length of the first row of $P(w)$

- The maximal length of a decreasing subword of w is the length of the first column of $P(w)$

We will prove something stronger
let $\ell_k(w)$ be the maximum of the length of k disjoint nondecreasing subwords of w.

$\ell_1(w) =$ maximal length of a nondecreasin subword of w

($\ell'_k(w)$ same for decreasing subwords)

ex 13254 $\quad \ell_1(w) = 3$, $\ell_2(w) = 5$, $\ell_3(w) = 5$, $\ell_4(w) = 5$

Thm 2 (Greene) For $w \in A^*$, If $P(w)$ has shape $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ & $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_s)$ then for $k = 1, \ldots, r$ $\quad \lambda_k = l_k(w) - l_{k-1}(w)$

$k = 1, \ldots, s$ $\quad \lambda'_k = l'_k(w) - l'_{k-1}(w)$

ex $w = 13254$, $P(w) = \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}$, $\lambda = (3,2)$, $\lambda' = (2,2,1)$, $l_1 = 3$, $l_2 = 3+2$, $l_3 = 5$, $l'_1 = 2$, $l'_2 = 4$, $l'_3 = 5$.

Thm 2 implies Thm 1.