HOMEWORK 1 (MATH 206, FALL 2014)

I. Consider alphabet $A = \{1, 2, 3\}$ and a sequence of words

$$w_0 \, = \, (13232)^\infty \, , \quad w_{k+1} \, = \, \vartheta(w_k),$$

where $\vartheta(u)$ is obtained by substituting w_0 into positions of 3's in u. For example,

$$w_1 \, = \, (1123212232122121322213222)^\infty .$$

(a) Prove that the limit word $w = \lim_{k\to\infty} w_k$ is well defined and contains no 3's.
(b) Prove that the word complexity $\varkappa(w, n) = \theta(n^\alpha)$ for some $\alpha \approx 3.15$.

II. Let $A = \{0, 1\}$, and consider a morphism $h(0) = 010$, $h(1) = 11$. Define a sequence of words $w_k \in A^*$ as follows:

$$(\circledast) \qquad w_1 = 0, \quad w_{k+1} = h(w_k).$$

For example,

$$w_2 = (010)(11)(010) \, , \quad w_3 = (010)(11)(010)(11)(11)(010)(11)(010).$$

(a) Prove that the limit word $w = \lim_{k\to\infty} w_k$ is well defined.
(b) Prove that the word complexity $\varkappa(w, n) = \theta(n \log \log n)$.

III. Let $A = \{0, 1, 2\}$, and consider a morphism $h(0) = 012$, $h(1) = 11$, $h(2) = 222$. Define a sequence of words $w_k \in A^*$ via $(\circledast)$. For example,

$$w_2 = 012 \, , \quad w_3 = (012)(11)(222) \, , \quad w_4 = (012)(11)(222)(11)(11)(222)(222)(222).$$

(a) Prove that the limit word $w = \lim_{k\to\infty} w_k$ is well defined.
(b) Prove that the word complexity $\varkappa(w, n) = \theta(n \log n)$.

IV. Let $A = \{0, 1\}$ and $w \in A^{100}$. Prove that there are two nonempty subwords $x, y \in A^*$ such that:
(a) x^2y^2 is a subword of w.
(b) x^2yx^2 is a subword of w.

V. Let $A = \{0, 1\}$ and two morphisms h_0, h_1 defined as follows:

$$h_0(0) = 0, \; h_0(1) = 00, \qquad h_1(0) = 1, \; h_1(1) = 11 \, .$$

Let $H : A^* \to A^*$ defined as follows:

$$H(x_1x_2x_3x_4 \ldots) \, = \, h_0(x_1)h_1(x_2)h_0(x_3)h_1(x_4) \ldots$$

Let $w_1 = 1$, $w_{k+1} = H(w_k)$, for all $k \geq 1$. For example,

$$w_2 = h_0(1) = 00, \;\; w_2 = h_0(0)h_1(0) = 01, \;\; w_3 = h_0(0)h_1(1) = 011, \;\; w_4 = h_0(0)h_1(1)h_0(1) = 01100.$$

(a) Prove that the limit word $w = \lim_{k\to\infty} w_k$ is well defined.
(b) Prove that w is cube free.

VI. Let w be a Sturmian word in the alphabet $A = \{0, 1\}$. Consider $w' = h(w)$, where $h(0) = 01$, $h(1) = 0$. Prove or disprove: w' is Sturmian.

VII. Let w be the Fibonacci word and let X^n be the set of subwords of w of length n. Prove that X^n contains exactly one palindrome for even n, and exactly two palindromes for odd n. For example, X^3 contains two palindromes (010) and (101), while X^4 contains only one palindrome (1001).

VIII. Let $w = (a_1 a_2 \ldots)$ be the Thue word. Prove that every finite word $u \in \{0,1\}^*$ is equal to an *arithmetic pattern* in w, i.e. of the form $(a_m a_{m+k} a_{m+2k} \ldots)$.

For example, recall that $w = (0110100110010110\ldots)$ contains 6 distinct subwords of length 3. The remaining 2 subwords of length 3 are given by arithmetic patterns $(000) = a_1 a_4 a_7$ and $(111) = a_2 a_5 a_8$.

IX. Consider n infinite word $w = (11011001110010011101100011001001\ldots)$ defined as a sequence $(a_1 a_2 \ldots)$ with $a_{4i} = 1$, $a_{4i+2} = 0$ and $a_{2i+1} = a_i$ for all $i \geq 0$.

(a) Prove that $\varkappa(w, n) = O(n)$.

(b) Prove that w has $O(n)$ arithmetic patterns of length n.

X. In the *Tower of Aleppo* game, there are three pegs: East, Center and West. Suppose there are n discs on the East peg. The goal is to move them to the West peg. However, only East-Center and Center-West moves are allowed. Find the shortest length of the solution.

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This Homework is due Fri November 7, 2014 before class. Collaboration is allowed, but only in groups of at most 3 students. In that case you must write the names of your collaborators on the front page of the solutions.