HOMEWORK 2 (MATH 184, WINTER 2018)

Read: Bona, sections 3.1-5. **Solve:**

I. Compute the closed form of (ordinary) g.f. for the following sequences:
a_n = number of tilings of $3 \times n$ rectangle with 1×3 bars (rotations allowed).
b_n = number of tilings of $3 \times n$ rectangle with L-shaped trominoes (rotations and reflections allowed).
c_n = number of spanning trees in a $2 \times n$ grid graph.
$d_n = F_1F_2 + F_2F_3 + \ldots + F_{n-1}F_n$, where F_n is the Fibonacci number.
e_n = number of proper 3-colorings of a $2 \times n$ grid (i.e. no two vertices of the same color can be connected by an edge).

II. Compute the explicit formulas for sequences in part I.

III. Compute the closed form of (ordinary) g.f. for the following number of partitions:
a_n = number of partitions of n with at most 3 parts and the smallest part even
b_n = number of partitions of n with exactly 9 parts, and the largest part 12.
c_n = number of partitions of n into distinct parts of size $\{1, 2, 4, 8, \ldots\}$.

IV. Let a_n be the set of partitions of n into parts $= 1$ or 5 mod 6. Let b_n be the set of partitions of n into distinct parts $\neq 0$ mod 3.
(1) Give a g.f. proof that $a_n = b_n$. For example, $a_6 = b_6$ since $a_2 = \#\{1^6, 51\} = 2$ and $b_2 = \#\{51, 42\} = 2$.
(2) Give a bijective proof of (1).

V. Let $q(n)$ denote the number of partitions of n into parts of size $\{1, 2, 4, 8, \ldots\}$.
(1) Prove that $q(n) = n^{\Omega(\log n)}$
(2) Prove that $q(n) = n^{O(\log n)}$

VI. Let $a(n, k)$ be the number of $\sigma \in S_n$ with $inv(\sigma) = k$. Prove that for every n the sequence $a(n, *)$ is *unimodal*:
$a(n, 0) \leq a(n, 1) \leq a(n, 2) \leq \ldots \leq a(n, \lfloor N/2 \rfloor) = a(n, \lceil N/2 \rceil) \geq \ldots \geq a(n, N-1) \leq a(n, N)$, where $N = \binom{n}{2}$.

⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆ ⋆

This Homework is due Wednesday February 28, at 2:59:59 pm. (right before class). Please read the collaboration policy on the course web page. Make sure you write your name in the beginning and your collaborators' names at the end.

P.S. All items above have the same weight.