MIDTERM (MATH 184, SPRING 2022)

Your Name: ________________________________ **(must be in ink)**

UCLA id: ________________________________ **(must be in ink)**

Date: __________________

The rules:

You MUST simplify completely and BOX all answers with an **INK PEN**.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Points:

1	
2	
3	
4	
5	

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Total: **(out of 100)**

Problem 1. (20 points)

Let $n = 10$. Compute the number of permutations $\sigma \in S_n$, such that:

a) $\sigma(2) - \sigma(4) = 3$

b) $\sigma(1) < \sigma(2)$ or $\sigma(2) > \sigma(3)$

c) σ has exactly two cycles

d) σ has exactly two inversions

Problem 2. (20 points)

George has 4 boxes and 10 tennis balls: 5 yellow and 5 green. George places all balls at random into boxes. Let $X =$ number of green balls in the 1st box, and $Y =$ number of yellow balls in the 1st box. Compute:

a) $\mathbb{P}(X = 0)$,
b) $\mathbb{P}(X = Y)$,
c) $\mathbb{E}[X^2 + Y^2]$,
d) $\mathbb{E}[X \cdot Y]$.

Problem 3. (20 points)

Let $a(n)$ be the number of partitions of n into distinct parts which are either 0, 1 or 2 mod 4. Let $b(n)$ be the number of partitions of n into parts which are either 1, 5 or 6 mod 8. For example, $a(5) = 2$ since $5 = 4 + 1$, and $b(5) = 2$ since $5 = 1 + 1 + 1 + 1 + 1$. Prove that $a(n) = b(n)$ for all $n \geq 1$.

Problem 4. (20 points)

Let $V = \mathbb{F}_q^n$ be a n-dimensional vector space space over the field with q elements. Let $e \in V$ be a nonzero vector. Compute the number of k-dimensional subspaces of V which do not contain e.

Problem 5. (20 points, 2 points each) **TRUE or FALSE?**

You MUST circle correct answer with INK. No explanation required.

T **F** (1) The number of permutations in S_n with k inversion = that with $\binom{n}{2} - k$ inversions.

T **F** (2) The number of k-subsets of n-set is equal to the number of (shortest) grid walks from $(0, 0)$ to (n, k).

T **F** (3) The number of integer partitions of n satisfies: $p(n) = \Omega(n^{\log n})$.

T **F** (4) $\binom{2n}{n} = \Theta(4^n)$.

T **F** (5) Stirling numbers $c(n, k)$ of the 1st kind count the number of permutations of $\{1, \ldots, n\}$ with k inversions.

T **F** (6) In the 100 prisoner's problem, if a warden allowed each prisoner opening 99 boxes, the prisoners can go free with probability $\geq 99\%$.

T **F** (7) In the 100 rabbits and 100 hunters problem, assume that it takes at least 2 hunters to kill each rabbit. Then, on average at least 80 rabbits will survive.

T **F** (8) Fixed points of Franklin's involution have Durfee square at most one.

T **F** (9) In class, we computed the variance of the Geo(p) random variable by induction.

T **F** (10) In class, we used a double counting argument to obtain a GF for the number of k-subsets with respect to inversions.