1.1. Cardinality of a set
1.2. Addition principle.
1.3. Generalized addition principle.

1.1. Cardinality of a set

Recall: given two sets $A$ and $B$, a map $f: A \to B$ is a subset $S$ of $A \times B$ such that for each $a \in A$ there is exactly one $b \in B$ such that $(a, b) \in S$.

One writes the elements of $S$ also as $(a, f(a))$.

A map $f: A \to B$ is:
- **injective** (or one-to-one) if for all $a, a' \in A$, $a \neq a'$ implies $f(a) \neq f(a')$.
- **surjective** (or onto) if for every $b \in B$ there is an $a \in A$ such that $f(a) = b$.
- **bijective** if it is injective and surjective.

Any injection $f: A \to B$ has a left-inverse, that is, there exists a map $g: B \to A$ such that $g \circ f = id_A$.

Similarly, any surjection has a right-inverse.

It follows that any bijection has both a left-inverse and a right-inverse.
and a right-inverse, and one can show that these coincide; thus any bijection has a two-sided inverse, which we denote by $f^{-1}$.

**Definition**

Let $n$ be a positive integer.

A finite non-empty set $A$ has **cardinality $n$** if there is a bijection $A \rightarrow \{1, \ldots, n\}$.

Notation: $[n] = \{1, \ldots, n\}$.

For the remainder of this week we will be concerned with studying some properties of the cardinality:

- $|A\cup B| = |A| + |B|$ \quad Addition principle (for $A \cap B = \emptyset$)
- $|A \setminus B| = |A| - |B|$ \quad Subtraction principle (for $B \subseteq A$)
- $|A \times B| = |A| \cdot |B|$ \quad Multiplication principle
- $|B|/|A| = \#_B$ \quad Division principle

If there is $f: A \rightarrow B$ 1-to-1

These equalities give different ways to count the elements of the same set, and we will see that sometimes one of the ways of counting is much easier than the other one.
1.2 Addition principle

Proposition

Let $A$ and $B$ be two disjoint, finite, non-empty sets.

Then $|A \cup B| = |A| + |B|$.

Proof

There exist positive integers $m$ and $n$ and bijections $f: [m] \to A$ and $g: [n] \to B$.

Our aim is to show that $|A \cup B| = m + n$, and for this we construct a bijection $[m+n] \to A \cup B$.

Define $h: [m+n] \to A \cup B$

$$x \mapsto \begin{cases} f(x), & 1 \leq x \leq m, \\ g(x-m), & m+1 \leq x \leq m+n \end{cases}$$

This is a bijection (ex.) $\Box$

1.3 Generalised addition principle

We want to extend the addition principle to any finite number of pairwise disjoint sets.
Recall: Mathematical induction

Let \( P(n) \) be a statement that involves a natural number \( n \in \mathbb{N} \). Let \( n_0 \in \mathbb{N} \).

If:
1) \( P(n_0) \) is true
2) For any \( k > n_0 \) assuming that \( P(k) \) is true it follows that \( P(k+1) \) is true

then \( P(n) \) is true for all \( n \geq n_0 \).

Proposition

Let \( A_1, \ldots, A_n \) be pairwise disjoint, non-empty and finite sets. Then \( |A_1 \cup \ldots \cup A_n| = |A_1| + \ldots + |A_n| \).

Proof

By induction.

\( n = 1 \) : there is nothing to prove.

Let \( k \) be a positive integer, and assume that the statement is true for \( k \).

Then
\[
|A_1 \cup \ldots \cup A_k \cup A_{k+1}| = |A_1 \cup \ldots \cup A_k| + |A_{k+1}|
\]

by the addition principle

\[
= (|A_1| + \ldots + |A_k|) + |A_{k+1}|
\]

by the induction assumption

\[ \square \]