Stacks from scratch

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> La conclusion pratique à laquelle je suis arrivé dès maintenant, c'est que chaque fois que en vertu de mes critères, une variété de modules (ou plutôt, un schéma de modules) pour la classification des variations (globales, ou infinitésimales) de certaines structures (variétés complètes non singulières, fibrés vectoriels etc) ne peut exister, malgré de bonnes hypothèses de platitude, propreté, et non singularité éventuellement, la raison en est seulement l'existence d'automorphismes de la structure qui empêche la technique de descente de marcher.
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> *Alexandre Grothendieck*
> *Paris 11.5.1959 [GCS04]*

Abstract

We give an introduction to stacks assuming a minimal background of algebraic geometry, consisting of varieties and elementary properties of schemes. We focus on motivating definitions and new constructions, and illustrate all new notions using the example of the stack of elliptic curves.

Introduction

Stacks were introduced in the 1960s to deal with issues arising in the study of moduli problems in algebraic geometry. A moduli problem is a classification problem: we have a class of objects, a notion of what it means for these objects to be equivalent and we would like to classify these equivalence classes. In these notes the objects will be elliptic curves and the equivalence relation will be given by isomorphisms. In a first encounter with elliptic curves, one defines an elliptic curve over a field. However, often we are more interested in studying not single mathematical objects, but rather how they vary in families. Families of elliptic curves are elliptic curves parametrised by a scheme. If one studies families of elliptic curves parametrised by a scheme S, then we can define a functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$$

which to every scheme assigns the set of isomorphism classes of elliptic curves parametrised by that scheme. This functor is the *moduli functor for elliptic curves.* We would then have a solution to the classification problem, or in other words, a *fine moduli scheme* of elliptic curves, if this functor were representable. However, due to the presence of non-trivial automorphisms of elliptic curves, the moduli functor for elliptic curves does not have a fine moduli scheme.

There are several solutions to this problem. One solution is given by the *coarse moduli scheme*, which we will discuss in Section 2. Other solutions are given by rigidifying the problem, namely by looking at objects with some extra structure, like marked points or level structures. Such moduli problems have been studied e.g. in [KM85].

These two approaches, even if useful, are however still unsatisfactory, since they forget some information about the structure of elliptic curves. One comes to a general solution if one observes that the source of trouble - namely the presence of non-trivial automorphism - is not a bug, but a feature of the moduli space of elliptic curves. The solution thus lies in considering not only the isomorphism classes of elliptic curves, but also their automorphisms. We thus pass from studying representability of the functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ to studying representability of the functor

$$\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Groupoid}$$

from the category of schemes to the category of groupoids, which to every family of elliptic curves over a scheme S assigns the category with objects isomorphism classes of elliptic curves and morphisms given by automorphisms.

The category of groupoids is not a category, but a 2-category, therefore making the previous statements precise involves translating the theory of schemes from the setting of categories into that of 2-categories. The 2-categorical generalisation of a sheaf is called stack, while the 2-categorical equivalent of a scheme is, roughly[1], an algebraic stack. Via the Yoneda lemma we can identify any scheme over a scheme S with the functor $\mathrm{Sch}/S^{\mathrm{op}} \to \mathrm{Set}$ that it represents. One way to pass from schemes to algebraic stacks is given by generalising the properties that a presheaf $\mathrm{Sch}/S \to \mathrm{Set}$ needs to

[1] An algebraic stack corresponds more closely to a 2-categorical version of an algebraic space, see Section 4.1.

satisfy to be a scheme, namely (i) that it is a sheaf (in the Zariski topology[2]) and (ii) that it is representable. The 2-categorical version of a scheme will therefore be a stack satisfying conditions analogous to those that a sheaf in the Zariski topology needs to satisfy to be representable. First, to develop the notion of stack (see Definition 3.26) one needs:

1. A more general notion of topology: Grothendieck topologies (see Section 3.1) generalise the Zariski topology, and one can define a more general notion of sheaves on such topologies
2. The 2-categorical equivalent of the gluing conditions for a presheaf to be a sheaf, which goes under the name of descent theory - (see Section 3.3, and Section 3.2 for fibered categories, which give the right framework to develop descent theory).

Second, we need to impose some conditions on stacks to be algebraic. A sheaf X in the Zariski topology is representable if the diagonal morphism $X \to X \times X$ is representable and if the sheaf can be covered by affine schemes (see, e.g. [Ols16, 1.4.13]). An algebraic stack is therefore a stack which satisfies conditions analogous to these (see Section 4).

Now we come back to our example of the moduli functor of elliptic curves: we will see in Example 3.17 that this functor is not a sheaf in the Zariski topology, and therefore there is no hope for it to be representable; however, the moduli stack of elliptic curves (see Examples 3.22 and 3.30) is a particularly nice kind of algebraic stack called Deligne-Mumford (DM) stack (see Theorem 4.10).

We conclude this introduction with the following diagram, which illustrates the zoo of geometric objects in algebraic geometry, of which we discuss but a few in this paper:

$$
\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\text{DM stacks} & \longrightarrow & \text{Alg. stacks} & \longrightarrow & \text{Stacks} & \longrightarrow & \text{Prestacks} & \longrightarrow & \text{Presh. of groupoids} \\
\uparrow & \nwarrow & & & \uparrow & & \uparrow & & \uparrow \\
\text{Sch}/S & \longrightarrow & \text{Alg. spaces} & \longrightarrow & \text{Sheaves} & \longrightarrow & \text{Sep. presh.} & \longrightarrow & \text{Presh. of sets}
\end{array}
$$

The geometric objects on the bottom level of the diagram are formulated in the theory of (1-) categories, while on the next level we need the formalism of 2-categories, and the n-th level from the bottom requires the formalism of n-categories. Finally, the colimit of this diagram requires the formalism of ∞-categories (see [Lur09]).

1 Fine moduli spaces

Let S be a scheme, and let $O(S)$ be a class of objects over S, which we construe as a 'family' of objects parametrized by S, that we want to classify, and suppose that we have a notion of what it means for two such objects to be equivalent. Such objects

[2] In fact, representable functors $\text{Sch}/S^{\text{op}} \to \text{Set}$ are sheaves in topologies finer than the Zariski topology (see Example 3.16).

could, e.g., be curves of genus g over S up to isomorphism, or closed subschemes of $\mathbb{P}^n_S$ up to Hilbert polynomial, and so on.

Ideally, we would like to have a geometric object whose points are in bijection with the equivalence classes of such objects, and such that its geometric structure gives information about how the objects vary in families. We define a functor

$$\mathrm{M}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$$

which to every scheme S assigns the set of equivalence classes of objects over S. We call such a functor a *moduli functor for O.*

By the Yoneda lemma we know that there is an embedding (i.e. a fully faithful functor) of the category of schemes Sch into the category of presheaves over Sch, that is, the category of functors $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$. Therefore we can ask when the moduli functor is in fact a geometric object, or in other words, when is it representable?

In the following we make the previous statements precise using the example of isomorphism classes of elliptic curves. We begin by recalling the definition of elliptic curves. One of the classic references for the theory of elliptic curves is [Sil86]. Throughout these notes we assume that k is an algebraically closed field.

Definition 1.1. *An* elliptic curve over k *is a smooth proper curve X over k which is connected and has genus 1 together with a rational point $O \in X$.*

To every elliptic curve (X, O) over k we can assign an element $j(X)$ of k called the *j-invariant of* (X, O). Two elliptic curves have the same j-invariant iff they are isomorphic, and furthermore for every $c \in k$ there is an elliptic curve (X, O) with $j(X) = c$. Therefore the set of isomorphism classes of elliptic curves is in bijection with the closed points of the affine line $\mathbb{A}^1_k$.

Usually one is not interested in studying moduli problems for individual curves, but rather in moduli spaces that capture how curves vary in 'families'. One therefore considers families of elliptic curves:

Definition 1.2. *Let k be an algebraically closed field, and let S be a scheme over k. An* elliptic curve over S *(also called a* family over S*) is a morphism of schemes $\pi\colon X \to S$ which is proper, smooth and such that the geometric fibers are connected curves of genus 1 together with a section $o\colon S \to X$. We denote an elliptic curve (X, O) over S by X/S if the section is clear from context.*

Note that an elliptic curve over $\mathrm{spec}(k)$ is the same thing as an elliptic curve over k, and therefore this definition generalises Definition 1.1. Furthermore, similarly to elliptic curves over fields, an elliptic curve over a scheme S has a natural structure of a group scheme over S, see e.g. [KM85].

Given an elliptic curve X/S and a morphism $S' \to S$, the pullback $X/S \times_S S'$ together with the induced section $O'\colon S' \to X/S \times_S S'$ is an elliptic curve X' over S'. One says that elliptic curves are 'preserved under base change'. This motivates the following definition:

Definition 1.3. *Given two elliptic curves X/S and X'/S' we define a* morphism of elliptic curves $X/S \to X'/S'$ *to be a pair of morphisms of schemes $f\colon X \to X'$ and $g\colon S \to S'$ such that the resulting square is cartesian.*

Now, given a scheme S, we denote by $\mathrm{M}_{1,1}(S)$ the set of isomorphism classes of elliptic curves over S. A morphism $S' \to S$ induces a map $\mathrm{M}_{1,1}(S) \to \mathrm{M}_{1,1}(S')$, and we thus get a contravariant functor $\mathrm{M}_{1,1}\colon \mathrm{Sch}/k \to \mathrm{Set}$ which we call the *moduli functor for elliptic curves.*

Definition 1.4. *Let $F\colon \mathrm{Sch}/k^{\mathrm{op}} \to \mathrm{Set}$ be a functor. A* fine moduli space *for F is a scheme M together with a natural isomorphism $F \to h_M$, where we denote by h_M the functor $\mathrm{Hom}_{\mathrm{Sch}}(-, M)\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ which sends a scheme N to the set $\mathrm{Hom}_{\mathrm{Sch}}(N, M)$.*

The moduli functor for elliptic curves does not have a fine moduli scheme. There are several ways to prove this. One way is given by the existence of non-trivial 'isotrivial families'.

Definition 1.5. *An elliptic curve X/S is* isotrivial *if all the fibers are isomorphic. An elliptic cuve X/S is* trivial *if it is the pullback of a family over a one-point scheme.*

If $\mathrm{M}_{1,1}$ were represented by a scheme M, then any isotrivial family would necessarily be trivial. To see this, let $\phi\colon \mathrm{M}_{1,1} \to h_M$ denote the natural isomorphism, and denote by U the elliptic curve over U given by $\phi_M^{-1}(id_M)$. Suppose that X/S is an isotrivial family; then X/S is the pullback of U/M along the morphism $S \to M$ which sends every $s \in S$ to a single point of M, and hence the family X/S is trivial. Now, there exist isotrivial families of elliptic curves which are not trivial, as for example if one considers the family given by the equation $y^2 = x^3 + t$ over $S = \mathrm{spec}(k[t, t^{-1}])$. For each t one gets a curve C with $j(C) = 0$, but there is no isomorphism between this family and the constant family $y^2 = x^3 + 1$, therefore the family is not trivial. Summing up, we have the following:

Theorem 1.6. *The moduli functor $\mathrm{M}_{1,1}$ does not have a fine moduli scheme.*

We will see in Example 3.17 that $\mathrm{M}_{1,1}$ fails to be a scheme more spectacularly.

2 Coarse moduli spaces

We have seen in the previous section that the functor $\mathrm{M}_{1,1}$ is not representable. In analogy with the case of elliptic curves over a field, we could ask for a scheme M such that for every elliptic curve X/S there is a morphism $\phi_X : S \to M$ such that for each closed point $s \in S$ the point $\phi_X(s)$ in M corresponds to the isomorphism class of the fiber over s in X. Furthermore, we could ask for the assignment $X/S \mapsto \phi_X$ to be functorial, namely we want this assignment to induce a natural transformation $\phi\colon \mathrm{M}_{1,1} \to h_M$. We would also like M to be unique up to a unique isomorphism satisfying some properties. More precisely, we can give the definition of coarse moduli space as follows:

Definition 2.1. *Let $F\colon \mathrm{Sch}/k^{\mathrm{op}} \to \mathrm{Set}$ be a functor. A* coarse moduli scheme *for F is a scheme M together with a natural transformation $\phi\colon F \to h_M$ and such that:*

1. *The map $\phi(\mathrm{spec}(k))$ is a bijection.*

2. *The scheme M satisfies the following universal property: whenever N is a scheme together with a natural transformation $\psi\colon F \to h_N$, then there exists a unique morphism of schemes $\overline{\psi}\colon M \to N$ such that $\phi \circ h_{\overline{\psi}} = \psi$.*

Note that every fine moduli scheme is a coarse moduli scheme: item (1) in the definition of coarse moduli scheme is satisfied since $\phi(S)$ is a bijection for every scheme S. To show that item (2) is satisfied it is enough to show that the natural transformation $h_M \cong F \to h_N$ is induced by a unique morphism $M \to N$. This follows from the Yoneda lemma, namely, from the fact that the functor $h\colon \mathrm{Sch}/k \to \mathrm{Fun}(\mathrm{Sch}/k^{\mathrm{op}}, \mathrm{Set})$ is fully faithful.

Now we prove that the moduli functor for elliptic curves has a coarse moduli space:

Theorem 2.2. *The affine line $\mathbb{A}^1_k$ is a coarse moduli space for* $\mathrm{M}_{1,1}$.

Proof sketch. First, for every elliptic curve X/S it is possible to define a morphism $S \to \mathbb{A}^1_k$ which assigns to a point s the j-invariant of the fiber over s. One can define locally over open affine subsets $\mathrm{spec}(R) \subset S$ an embedding of the fiber over $\mathrm{spec}(R)$ into $\mathbb{P}^2_R$. The image of the fiber can be brought into a certain Weierstrass form, using which one can define a j-function on $\mathrm{spec}(R)$. Gluing gives a morphism $\phi_S\colon S \to \mathbb{A}^1_k$ with the property that for every closed point $s \in S$ the image $\phi_S(s)$ is the j-invariant of the fiber over s. For more details we refer the reader to [KM85] or [Del].

We already know from the theory of elliptic curves over fields that the closed points of $\mathbb{A}^1_k$ are in bijection with the isomorphism classes of elliptic curves over k. This in turn implies that $\phi(\mathrm{spec}(k))\colon F(\mathrm{spec}(k)) \to h_M(\mathrm{spec}(k))$ is a bijection.

It remains to show that $\mathbb{A}^1_k$ satisfies the universal property of a coarse scheme. For this, let N be a scheme and $\psi\colon F \to h_N$ a natural transformation. Then one can construct a morphism $\mathbb{A}^1_k \to N$ in the following way: consider the family of elliptic curves given by the equation $y^2 = x(x-1)(x-\lambda)$ over $\mathrm{spec}(k[\lambda, \lambda^{-1}, (\lambda-1)^{-1}])$. We denote the ring $k[\lambda, \lambda^{-1}, (\lambda-1)^{-1}]$ by R. Then one shows that the morphism $\psi(\mathrm{spec}(R))\colon \mathrm{spec}(R) \to N$ factors through $\mathrm{spec}(R^{\Sigma_3})$, where Σ_3 is the symmetric group of order 6 acting on R and R^{Σ_3} is the ring of invariants under this action. One further shows that $\mathbb{A}^1_k$ can be identified with $\mathrm{spec}(R^{\Sigma_3})$ and thus we obtain a morphism $\mathbb{A}^1_k \to N$. Uniqueness of this morphism follows by construction.

□

We have seen that every fine moduli scheme is a coarse moduli scheme. On the other hand, suppose that we are given a coarse moduli scheme M. Then if we can pull back any scheme X over S along a 'universal' family U over M, the moduli space is fine.

More precisely, we give the following definition:

Definition 2.3. *Let $F\colon \mathrm{Sch}/k^{\mathrm{op}} \to \mathrm{Set}$ be a functor. If a coarse moduli space M for F exists, then a* universal family over M for F *is a scheme U over M such that for any scheme X over S there is a unique morphism $S \to M$ such that X is isomorphic to the pullback $S \times_M U$.*

If such a scheme M and universal family U/M exist, then the functor F is represented by M; in other words, the natural transformation $\phi\colon F \to h_M$ is an isomorphism. The converse is also true: if F is represented by a scheme M, then the scheme

$\phi_M^{-1}(id_M)$ is a universal family over M for F. Therefore a fine moduli scheme is the same thing as a coarse moduli scheme together with a universal family.

3 Stacks

As we have seen in the previous sections, not every moduli functor is representable. The problem, as in the case of elliptic curves, is often given by objects with non-trivial automorphisms. As mentioned in the introduction, the solution lies in considering not functors valued in sets, but in groupoids. Any scheme can be identified with a functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ that is a sheaf in the Zariski topology. To generalise this, we will on one hand need a more general definition of topology, and on the other hand a more general notion of sheaf, or in other words, of gluing. Grothendieck topologies generalise the Zariski topology, while the gluing condition for sheaves is generalised by the theory of descent. The right framework to develop descent theory is given by categories fibered in groupoids. We can then say that, roughly, a stack is a functor $\mathrm{Sch}^{\mathrm{op}} \to \mathrm{Groupoid}$ satisfying gluing conditions similar to those of a sheaf, with the difference that while a sheaf takes values in a 1-category, a stack takes values in a 2-category. This generalisation is not gratuitous: there are stacks, e.g. the stack of elliptic curves, which are not sheaves.

3.1 Grothendieck topologies

In this section we introduce the definition of a Grothendieck topology using sieves, as it is given e.g. in [GV72].[3]

A sieve on an object U of a category C can be thought of as a choice of morphisms to U that is closed under precomposition. A Grothendieck topology on C is the assignment of a collection of sieves to every object, so that these collections of sieves satisfy some conditions, like e.g. being closed under 'pullbacks'. A site is a category together with a Grothendieck topology, and a pretopology is the analogous of a basis for a topology in the classical sense. After we develop the basic facts about Grothendieck topologies, we introduce sheaves on sites in Section 3.1.1. These are Set-valued contravariant functors on a site satisfying a gluing condition which generalises the gluing conditions for a presheaf to be a sheaf in the Zariski topology. We end this section with two important examples: schemes are sheaves in Grothendieck topologies finer than the Zariski topology (see Example 3.16); the moduli functor of elliptic curves is not a sheaf (see Example 3.17).

For this section we follow the exposition in [MM92].

Definition 3.1. *A* sieve of a category *C is a full subcategory S of C such that for any $U \in \mathrm{ob}S$ and for any morphism $f\colon U' \to U$ of C the object U' is in S. A* sieve of an object *U* of *C is a sieve of the category C/U.*

Sieves can be seen as a generalization of ideals:

[3] Note that some authors (see e.g. [FGI$^+$05, Part 1]) call Grothendieck topology what is usually known as a Grothedieck pretopology (see Definition 3.8) or basis.

Example 3.2. *Suppose that C has just one object, so that it is a monoid. Then the sieves on C are its right ideals.*

One can also construe sieves on an object $U \in \mathrm{ob}C$ as subfunctors of $\mathrm{Hom}_C(-, U)$.

Definition 3.3. *A* subfunctor *of a functor $F \colon C \to \mathrm{Set}$ is a functor $F' \colon C \to \mathrm{Set}$ such that:*

- *For any $U \in \mathrm{ob}C$ we have $F'(U) \subset F(U)$.*
- *For any $f \colon U' \to U$ the morphism $F'(f)$ is the restriction of $F(f)$ to $F'(U')$.*

Proposition 3.4. *A sieve on $U \in \mathrm{ob}C$ is a subfunctor of $\mathrm{Hom}_C(-, U)$.*

If we are given a sieve S on an object U and a morphism $f \colon U' \to U$, then the set

$$f^\star(S) = \{g \mid gf \in S\}$$

is a sieve on U'.

Definition 3.5. *Let C be a small category. A* topology *J on C is the assignment to each object U of C of a set $J(U)$ of sieves on U which satisfies the following:*

(i) *For any object U of C the maximal sieve $S_{\max} = \{f \mid \mathrm{cod} f = U\}$ is in $J(U)$.*

(ii) *For any object U of C for any sieve $S \in J(U)$ and for any morphism $f \colon U' \to U$ in C the sieve $f^\star(S)$ is in $J(U')$.*

(iii) *If R is any sieve on an object U such that there exists a sieve $S \in J(U)$ so that for all morphisms $f \colon U' \to U$ in S the sieve $f^\star(R)$ is in $J(U')$, then R is in $J(U)$.*

Definition 3.6. *A* site *is a tuple (C, J) consisting of a small category C together with a Grothendieck topology J on it.*

Example 3.7. *Let X be a topological space and denote by $O(X)$ the category with objects the open subsets of X and morphisms the inclusions. Then a sieve on an object U of $O(X)$ is a set S of open subsets of U such that $V' \subset V \in S$ implies that $V' \in S$. A Grothendieck topology on $O(X)$ is given by the assignment to every open subset U of X of a covering $\{V_i\}_{i \in I}$, i.e. a set of open subsets of U such that for any point x in U there exists an i such that $x \in V_i$.*

Note that not every cover of an open subset is a sieve. However, to any cover $\{U_i\}_{i \in I}$ of an open subset U we can associate a unique sieve generated by it: it is given by the set of all open subsets V of U such that there exists an $i \in I$ with $V \subset U_i$. This process of generation of a sieve can be generalized to arbitrary categories:

Definition 3.8. *A* pretopology *on a category C is the assignment to each object U of C of a set $\mathrm{Cov}(U)$ consisting of sets of morphisms with codomain U called* coverings of U *such that:*

(i′) *If $f \colon V \to U$ is an isomorphism then $\{f \colon V \to U\}$ is in $\mathrm{Cov}(U)$.*

(ii′) *If $\{f_i \colon U_i \to U \mid i \in I\}$ is in $\mathrm{Cov}(U)$ and $f \colon V \to U$ is any arrow, then for every $i \in I$ the fibered product $(U_i \times_U V, \pi_{U_i}, \pi_V)$ exists and the collection of morphisms $\{\pi_V \colon U_i \times_U V \to V \mid i \in I\}$ is in $\mathrm{Cov}(V)$.*

(iii′) *If* $\{U_i \to U \mid i \in I\}$ *is in* $\mathrm{Cov}(U)$ *and for any* $i \in I$ *there is a set* $\{f_{ij} \colon V_{ij} \to U_i \mid j \in J_i\} \in \mathrm{Cov}(U_i)$ *then* $\{f_i \circ f_{ij} \colon V_{ij} \to U \mid i \in I, j \in J_i\}$ *is in* $\mathrm{Cov}(U)$.

Note that in the original definition [**?**] instead of condition (i') there was the more restrictive condition

(i'') $id_U \colon U \to U$ is in $\mathrm{Cov}(U)$,

however this is not a real difference if we restrict our attention to topologies. Clearly (i') implies (i''). If we consider a family satisfying (ii'), (iii') and (i''), then it generates a pretopology. And it turns out that the topologies generated by the pretopology and by the original family are the same, since the sieve generated by an isomorphism $Y \to X$ is $Hom(-, X)$.

Because of condition (i'), a topology is not a pretopology, however any pretopology generates a unique topology:

Proposition 3.9. *Given a pretopology* Cov *on a small category* C*, it generates a unique topology in the following way*

$$J \colon U \mapsto \{S \in \mathrm{Siev}(U) \mid \exists R \in \mathrm{Cov}(U) \text{ such that } R \subset S\}$$

where we denote by $\mathrm{Siev}(U)$ *the set of sieves on* U*.*

Proof. We first verify that for every object U the maximal sieve is in $J(U)$: for this it is enough to note that if $f \colon U' \to U$ is an isomorphism, then the sieve generated by $\{f \colon U' \to U\}$ is the functor $Hom_C(-, U)$. For condition (ii) in the definition of a topology, let $S = \{f_i \colon U_i \to U \mid i \in I\} \in J(U)$ and let $f \colon U' \to U$ be a morphism in C. By (ii') the set $T = \{\pi_d : U \times_U U_i \to U' \mid i \in I\}$ is a cover of U'. Since $f \circ \pi_{U'} = \pi_{U_i} \circ f_i$ for all $i \in I$, we have that $T \subset f^\star(S)$, hence $f^\star(S)$ is in $J(U')$. For the transitivity condition, i.e. condition (iii), assume that $S \in J(U)$ and for a sieve R on U and that for any morphism $f \colon U' \to U$ that $f^\star(R) \in J(U')$. Let T be a covering contained in S and for $f \in T$ let T' be a covering contained in $f^\star(R)$. Then by (iii') the collection of arrows $T'' = \{h \circ f \mid h \in T', f \in T\}$ is a covering of U and since $T'' \subset R$, we have that R is in $J(U)$. ☐

Example 3.10. *[FGI+ 05, Part 1] A pretopology on* $O(X)$ *is given by assigning to any open subset* U *of* X *the set of open covers of* U*.*

Example 3.11. *Let a covering for any topological space* U *be given by a family* $T = \{f_i \colon U_i \to U \mid f_i \text{ is injective, open continuous}\}$ *such that* T *is jointly surjective, meaning that* $\sqcup U_i \to U$ *is surjective. To see that this defines a pretopology on* Top*, it is enough to note that in a pulback square of topological spaces*

$$\begin{array}{ccc} U_i \times Y & \longrightarrow & Y \\ {\scriptstyle \pi_{U_i}} \downarrow & & \downarrow {\scriptstyle f} \\ U_i & \longrightarrow & U \end{array}$$

f *is surjective iff* p *is.*

Similarly to classical topologies, there is a partial order on Grothendieck topologies:

Definition 3.12. *Given two Grothendieck topologies J and J' on a category C, we say that J is* coarser *than J' and J'* finer *than J if every sieve in J is also in J'.*

The coarsest possible topology is clearly the topology given by assigning to each object U the one-point set $J(U) = \{\mathrm{Hom}_C(-, U)\}$, whilst the finest topology is given by the assignment $J(U) = \mathrm{Siev}(U)$.

Examples 3.13. *Let S be a scheme. We give some examples of pretopologies on* Sch/S.

(i) Zariski (pre)topology: the coverings of any scheme X over S are sets of jointly surjective open embeddings.

(ii) Étale (pre)topology: the coverings of any scheme X over S are sets of jointly surjective étale morphisms.

(iii) fppf (pre)topology: the coverings of any scheme X over S are sets of jointly surjective flat and locally finitely presented morphisms.

Note that the Zariski topology is coarser than the étale topology, which is coarser than the fppf topology.

3.1.1 Sheaves on a site

We are now ready to give a more general definition of sheaf:

Definition 3.14. *A* sheaf *on a site (C, J) is a presheaf F on C such that for any object U in C and any $S \in J(U)$ the diagram*

$$F(U) \xrightarrow{e} \prod_{f \in S} F(\mathrm{dom} f) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\substack{f, f' \\ f \in S \\ \mathrm{dom} f = \mathrm{cod} f'}} F(\mathrm{dom} f')$$

is an equalizer of sets, where the maps are defined as follows:

- *e sends $x \in F(U)$ to $x_f := F(f)(x)$ in the component $F(\mathrm{dom} f)$ of the product*
- *a sends x_f to $x_{f \circ f'}$ in the component $F(\mathrm{dom} f')$ of the product (recall that if $f \in S$ and f and f' are composable, then $f \circ f' \in S$.)*
- *p send x_f to $F(f')(x_f)$ in the component $F(\mathrm{dom} f')$ of the product.*

Here we give an equivalent formulation in terms of a pretopology:

Proposition 3.15. *Let C be a category with pullbacks, let* Cov *be a pretopology on C and J the unique topology on C generated by* Cov*. A presheaf F on C is a sheaf on (C, J) iff for any object U in C and any cover $\{U'_i \to U\}_{i \in I}$ of U, the following diagram*

$$F(U) \xrightarrow{e} \prod_{i \in I} F(U'_i) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \underset{i,j \in I}{F} (U'_i \times_U U'_j)$$

is an equalizer, where the map e is defined as above, and the maps p_1 and p_2 are induced by the projections $U'_i \times_U U'_j \to U'_i$ and $U'_i \times_U U'_j \to U'_j$.

Example 3.16. *Representable functors* Sch$/S \to$ Set *are sheaves in the fppf topology [Gro71], and therefore also in all coarser topologies. (See also [FGI$^+$05, Part 1] for a discussion.)*

Example 3.17. *The moduli functor for elliptic curves is not a sheaf in the Zariski topology (and thus it is not a sheaf in any finer topology). A way to see this is to construct two families of curves over a scheme S that are not isomorphic but become isomorphic on some Zariski open subset U of S. For example, let E be an elliptic curve over k, and consider the constant family $E \times_k \mathbb{P}^1_k$. Let $p \neq q \in \mathbb{P}^1_k$ and set $S = \mathbb{P}^1_k/p \sim q$. Define a non-trivial elliptic curve $E' \to S$ by identifying the fibers over p and q with a non-trivial automorphism. Then the resulting family $X \to S$ is not isomorphic to the constant family $E \times_k S \to S$, however their restrictions to the open subscheme $S \setminus \{[p]\}$ are isomorphic.*

Note that the previous two examples give another proof of the fact that the moduli functor for elliptic curves is not representable.

3.2 Fibered categories

Let $\mathcal{F}$ and C be categories, and $p\colon \mathcal{F} \to C$ a functor. Given an object U in C, we want to consider the *fiber* $\mathcal{F}(U)$ of $\mathcal{F}$ over U, which is the category with objects $\eta \in F$ such that $p(\eta) = U$ and morphisms $\alpha\colon \eta \to \eta'$ such that $p(\alpha) = id_U$. If we allow F to be any category, then the notion of fiber would not be very useful (i.e. not functorial), since we could have objects $U \cong V$ in C such that $F(U)$ is empty while $\mathcal{F}(V)$ is not. It turns out that this notion is sensible whenever $\mathcal{F}$ is a *fibered* category. Roughly, $\mathcal{F}$ is a fibered category if every morphism in C has a universal lift in $\mathcal{F}$: given a morphism $f\colon T \to S$ in C and an object $\zeta \in F$ such that $p(\zeta) = S$, we say that there exists a 'lift' of the morphism f to the category $\mathcal{F}$ if there exists a morphism α in $\mathcal{F}$ with codomain ζ and such that $p(\alpha) = f$. Such a lift might not exist, or there might be many different morphisms giving a lift. When every morphism in C has a universal lift in $\mathcal{F}$, then we say that $\mathcal{F}$ is *fibered over C*. In the following we will make the previous statements precise.

Definition 3.18. *Let $\mathcal{F}$ and C be categories, and $p\colon \mathcal{F} \to C$ a functor. We say that $\mathcal{F}$ is a* category over C*. Similarly, if $\eta \in \mathcal{F}$ is such that $p(\eta) = T$, then we say that η is an* object of $\mathcal{F}$ over T*. An arrow $\alpha\colon \eta \to \zeta$ is* cartesian *if given $T = p(\eta)$, $S = p(\zeta)$ and $f = p(\alpha)$, for all $\eta' \in \mathcal{F}$ over T and for all $u\colon \eta' \to \zeta$ such that $p(u) = f$ there is exactly one morphism $\overline{u}\colon \eta' \to \eta$ such that $\alpha \circ \overline{u} = u$. For such a cartesian arrow α we also say that η is a* pullback of ζ over T.

The definition of cartesian arrow given in the previous definition was introduced in [Gro71]. Nowadays, a cartesian arrows is usually defined by a stronger universal condition, see e.g. [FGI$^+$05, Part 1]. However, in a fibered category the two notions are equivalent.

Definition 3.19. *A category $\mathcal{F}$ over C is* fibered *if for every morphism $f\colon U \to V$ and any object η of $\mathcal{F}$ over V, there is a cartesian arrow $\phi\colon \zeta \to \eta$ such that ϕ gets mapped to f. Let $\pi'\colon \mathcal{F}' \to C$ and $\pi\colon \mathcal{F} \to C$ be fibered categories over C. A* morphism of fibered categories over C *is a functor $\psi\colon \mathcal{F}' \to \mathcal{F}$ such that:*

(i) $\pi' = \pi \circ \psi$

(ii) ψ *sends cartesian arrows to cartesian arrows.*

Now, given a fibered category $p\colon \mathcal{F} \to C$, for any $U \in C$ one can define the *fiber of* $\mathcal{F}$ *over* U to be the category with the following:

- objects: objects of $\mathcal{F}$ over U
- morphisms: those morphisms of $\mathcal{F}$ that get sent to id_U.

Given a fibered category $\mathcal{F} \to C$, and for any arrow $f\colon U \to V$ in C and object η of $\mathcal{F}$ over V a choice of cartesian arrow $f^\star\colon f^\star\eta \to \eta$ such that $p(f^\star) = f$, it is possible to define a functor $f^\star\colon \mathcal{F}(V) \to \mathcal{F}(U)$. This motivates the definition of *cleavage*, which is just a choice of pullback:

Definition 3.20. *Given a fibered category* $p\colon \mathcal{F} \to C$*, a* cleavage of F *is a set* $\mathcal{C}$ *of cartesian morphisms of* F *such that for any morphism* $f\colon T \to S$ *in* C *and any* η *over* S*, there exists exactly one morphism* α *in* $\mathcal{C}$ *with codomain* η *and such that* $p(\alpha) = f$*.*

To recap, if we have a fibered category $\mathcal{F}$ over C and a cleavage of F, then for every object T of C we can define a category $\mathcal{F}(T)$ and for any morphism $f\colon T \to S$ a functor $f^\star\colon \mathcal{F}(T) \to \mathcal{F}(S)$. This assignment induces a pseudofunctor F from the category C^{op} to Cat, that is, a 'functor' from the 1-category C^{op} to the 2-category Cat which preserves composition and identity not up to equality, but up to a natural isomorphism. We refer the reader to [FGI+05, 3.1.2] for details.

On the other hand, given a category C and a pseudofunctor $F\colon C^{\mathrm{op}} \to \mathrm{Cat}$, one can assign to it a fibered category $\mathcal{F}$ over C together with a cleavage: an object of $\mathcal{F}$ is given by a pair (S, η) where $S \in C$ and $\eta \in F(S)$, and a morphism $(S, \eta) \to (T, \zeta)$ in $\mathcal{F}$ consists of a pair (f, α) where $f\colon S \to T$ is an arrow in C and $\alpha\colon \eta \to \mathcal{F}(f)(\zeta)$ is a morphism in $F(S)$. For more details, see [FGI+05, 3.1.3].

In practice, one is often interested in fibered categories where all fibers are sets, or groupoids. We therefore define the following:

Definition 3.21. *A category* $\mathcal{F}$ *over* C *is* fibered in groupoids *(resp.* sets*) if it is fibered over* C *and all fibers are groupoids (resp.* sets*).*

Example 3.22. *(The fibered category of elliptic curves.) Let* $\mathcal{M}_{1,1}$ *be the category with objects elliptic curves, and morphisms morphisms of elliptic curves (see Defs. 1.2-1.3). Let* $p\colon \mathcal{M}_{1,1} \to \mathrm{Sch}/k$ *be the functor that sends a family over* S *to* S*. Then* $\mathcal{M}_{1,1}$ *is fibered in groupoids over* Sch/k*: let* $f\colon S' \to S$ *be a morphism of schemes, and let* $\pi\colon Y \to S$ *be a family over* S*. Consider the pullback* $(Y \times_S S', \overline{f}, \overline{\pi})$*. Then the morphism* $\overline{f}$ *is cartesian from general principles (namely by the universal property of the pullback). Similarly, the fact that for any scheme* S *the fiber* $\mathcal{M}_{1,1}(S)$ *is a groupoid follows from general principles.*

We end this section with some remarks on properties of categories fibered in sets, which we will need in the last section. There is an equivalence of categories between the category of categories fibered in sets over C and the category of presheaves on C [FGI+05, 3.26]. In particular, if S is a scheme, we can associate to the presheaf $h_S\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$ the category fibered in sets $\mathcal{H}_S \to \mathrm{Sch}$ defined as follows:

- objects: pairs (U, f) where U is a scheme and $f \in h_S(U)$

- morphisms: commutative triangles.

On the other hand, the category Sch/S is also a category fibered in sets over Sch, and it is isomorphic over Sch to $\mathcal{H}_S$. We can therefore identity a scheme S with the fibered category $\mathrm{Sch}/S \to \mathrm{Sch}$.

3.3 Descent and stacks

Recall from the previous section that given a fibered category $\mathcal{F}$ over C together with a cleavage, then for any morphism $f\colon S' \to S$ in C, we have a functor $f^\star\colon \mathcal{F}(S) \to \mathcal{F}(S')$. The basic aim of descent theory is to characterise the image of the functor $f^\star$.

The solution given by Grothendieck is to take $S'' = S' \times_S S'$ and consider the two projections $p_1, p_2\colon S'' \rightrightarrows S'$. A *covering datum* for an object η of $\mathcal{F}$ over S' is an isomorphism $p_1^\star(\eta) \to p_2^\star(\eta)$. One can then define a category of objects of $\mathcal{F}(S')$ together with covering data with respect to the two projections. The functor $f^\star$ factors canonically through this category. Further, one defines a *descent datum* for an object η of $\mathcal{F}$ over S' to be a covering datum that satisfies a certain cocycle condition, and one sees again that the functor $f^\star$ factors through the full subcategory of the category of covering data given by objects with descent data.

The problem of descent is thus reduced to the following question: when is the functor $f^\star\colon \mathcal{F}(S) \to \mathcal{F}(S')_{\mathrm{desc}}$ an equivalence?

In the following we will make all of this precise. The diagram that one should keep in mind is the following:

$$\cdots \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; S' \times_S S' \times_S S' \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; S' \times_S S' \rightrightarrows S' \xrightarrow{f} S.$$

The basic idea of descent is that objects in the essential image of the functor $f^\star$ satisfy some coherence condition described by the simplicial diagram on the left-hand side. For the theory of 1-stacks it is enough to consider only the coherence conditions given by the pullback $S' \times_S S'$ and the iterated pullback $S' \times_S S' \times_S S'$, while for higher stacks one considers also higher coherence conditions given by higher iterations of the pullback.

Descent theory was first introduced in [Gro60], and this is also the main reference for the material in this section.

Definition 3.23. *Let $\mathcal{F}$ be a category fibered over C together with a cleavage. Let $\beta_1, \beta_2\colon S'' \to S'$ be two morphisms in C, and let ζ be an object in $\mathcal{F}(S')$. A* covering datum for ζ with respect to (β_1, β_2) *is an isomorphism $\beta_1^\star(\zeta) \to \beta_2^\star(\zeta)$.*

One can define a category $\mathcal{F}(S')_{\mathrm{cov}}$ of covering data with respect to (β_1, β_2):

- its objects are pairs (ζ, ϕ) where $\zeta \in \mathcal{F}(S')$ and ϕ is a covering datum for ζ
- a morphism $(\zeta, \phi) \to (\zeta', \phi')$ is a pair $(f\colon \beta_1^\star(\zeta) \to \beta_1^\star(\zeta'), g\colon \beta_2^\star(\zeta) \to \beta_2^\star(\zeta')$ of morphisms in $\mathcal{F}(S'')$ such that $f \circ \phi' = \phi \circ g$.

Furthermore, to any morphism $\alpha\colon S' \to S$ such that $\alpha \circ \beta_1 = \alpha \circ \beta_2$ we can associate a canonical functor $\mathcal{F}(S) \to \mathcal{F}(S')_{\mathrm{cov}}$ in the following way:

- to any object η over S we can associate the object $\alpha^\star(\eta)$ over S', which has a canonical covering datum since $\beta_i^\star\alpha^\star(\eta) \cong (\alpha\beta_i)^\star(\eta)$, and $\alpha\beta_1 = \alpha\beta_2$ by assumption.
- to any morphism $f\colon \eta \to \zeta$ in $\mathcal{F}(S)$ we can assign the morphism $\alpha^\star(f)\colon \alpha^\star(\eta) \to \alpha^\star(\zeta)$ which is compatible with the covering data.

Usually, one considers $S'' = S' \times_S S'$. In this case one has that an object in the essential image of $f^\star$ necessarily satisfies the following *cocycle condition*:

$$\begin{array}{ccccc} p_{12}^\star p_1^\star & \xrightarrow{p_{12}^\star\phi} & p_{12}^\star p_2^\star & = & p_{23}^\star p_1^\star \\ \| & & & & \downarrow{\scriptstyle p_{23}^\star\phi} \\ p_{13}^\star p_1^\star & \xrightarrow{p_{13}^\star\phi} & p_{13}^\star p_2^\star & = & p_{23}^\star p_2^\star \end{array}$$

where we denote by the equal sign the canonical isomorphisms given by the pullbacks. This motivates the following definition:

Definition 3.24. *Let (ζ, ϕ) be an object in $\mathcal{F}(S')_{\mathrm{cov}}$. We say that ϕ is a* descent datum *if $p_{23}^\star\phi \circ p_{12}^\star\phi = p_{13}^\star\phi$.*

We can summarise the previous discussion with the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{f^\star} & \mathcal{F}(S') \\ \downarrow & & \uparrow \\ \mathcal{F}(S')_{\mathrm{desc}} & \longrightarrow & \mathcal{F}(S')_{\mathrm{cov}} \end{array}$$

The morphisms f that we are most interested in are those for which $f^\star$ is an equivalence, and they thus get their own name:

Definition 3.25. *One calls a morphism f an* effective descent morphism *if $f^\star\colon \mathcal{F}(S) \to \mathcal{F}(S')_{\mathrm{desc}}$ is an equivalence.*

Now, if C is a site, what one is really interested in is usually not a morphism $f\colon S' \to S$, but a covering $\{f_i\colon S_i' \to S\}_{i\in I}$ of an object S in the site C. One can generalise what we have discussed to this setting: the category $\mathcal{F}(S_i', i \in I)_{\mathrm{cov}}$ has as objects pairs $(\{S_i'\}_{i\in I}, \{\phi_{i,j}\}_{i,j\in I})$ where for any pair of objects ζ_i' in $\mathcal{F}(S_i')$ and ζ_j' in $\mathcal{F}(S_j')$ we have that $\phi_{i,j}\colon p_1^\star(\zeta_i') \to p_2^\star(\zeta_j')$ is an isomorphism in $\mathcal{F}(S_i \times_S S_j')$.

Furthermore, one generalises the cocycle condition by requiring that the following diagram commutes in $\mathcal{F}(S_i \times_S S_j \times_S S_k)$:

$$\begin{array}{ccccc} p_{12}^\star p_1(\zeta_i)^\star & \xrightarrow{p_{12}^\star\phi} & p_{12}^\star p_2^\star(\zeta_j) & = & p_{23}^\star p_1^\star(\zeta_j) \\ \| & & & & \downarrow{\scriptstyle p_{23}^\star\phi} \\ p_{13}^\star p_1^\star(\zeta_i) & \xrightarrow{p_{13}^\star\phi} & p_{13}^\star p_2^\star(\zeta_k) & = & p_{23}^\star p_2^\star(\zeta_k) \end{array}$$

Similarly to the case of one morphism, there is a functor $f^\star\colon \mathcal{F}(S) \to \mathcal{F}(S_i', i \in I)_{\mathrm{desc}}$, and one says that the covering $\{f_i\colon S_i' \to S\}_{i\in I}$ is *of effective descent* if the functor $f^\star$ is an equivalence.

We can now give the following definition:

Definition 3.26. *Let C be a site. Let $\mathcal{F}$ be a category fibered in groupoids over C together with a cleavage. Then $\mathcal{F}$ is a* stack *if for every $S \in C$ and any covering $\{S_i' \to S\}_{i\in I}$ of S, the functor $\mathcal{F}(S) \to \mathcal{F}(S_i, i \in I)_{\text{desc}}$ is an equivalence.*

In nice cases we only need to prove descent for coverings consisting of a single morphism:

Lemma 3.27. *[Ols16, Lemma 4.2.7]. Let C be a site in which coproducts exist, and such that coproducts commute with fiber products when they exist. Let $\mathcal{F}$ be a category fibered over C, and assume that there is an equivalence*

$$\mathcal{F}\left(\bigsqcup_i U_i\right) \to \prod_i \mathcal{F}(U_i)$$

Let $\{S_i' \to S\}_{i\in I}$ be a collection of morphisms in C. Then

$$\mathcal{F}(S) \to \mathcal{F}(S_i', i \in I)_{\text{desc}}$$

is an equivalence iff

$$\mathcal{F}(S) \to \mathcal{F}\left(\bigsqcup_i S_i'\right)_{\text{desc}}$$

is.

As a first example of descent, we consider sheaves on a site. Let C be a site. For any object S in C, denote by $\text{Sh}(X)$ the category of sheaves on the site C/S. Any morphism $f: S' \to S$ induces a functor $f^\star: \text{Sh}(S) \to \text{Sh}(S')$ in a natural way (see [Ols16, 4.2.11] for details). Let Sh be the following category:

(i) objects: pairs (S, F) where $S \in C$ and $F \in \text{Sh}(S)$

(ii) morphisms: pairs $(f, \alpha): (S', F') \to (S, F)$ where $f: S' \to S$ is a morphism in C and $\alpha: F' \to f^\star(F)$ is a morphism in $\text{Sh}(F')$.

The assignment $(S, F) \mapsto F$ induces a functor $p: \text{Sh} \to C$ which makes Sh into a category fibered over C with fiber over an object S given by the category $\text{Sh}(S)$ of sheaves on Sch/S. Then one has the following result:

Theorem 3.28. *[Ols16] For any object Y in C and any cover $f: U \to Y$, the functor $f^\star: Sh(Y) \to Sh(U)_{\text{desc}}$ is an equivalence.*

To prove descent for elliptic curves, we follow [Ols16]. A *polarized scheme* is a pair $(f: X \to Y, L)$ where f is a proper and flat morphism of schemes and L is a relatively ample invertible sheaf on X. Morphisms of polarized schemes are cartesian squares. Let Pol denote the category of polarized schemes. The functor

$$\text{Pol} \to \text{Sch}$$

sending a pair $(f: X \to Y, L)$ to Y makes Pol a fibered category. One has the following:

Theorem 3.29. *For any covering $S' \to S$ in the fppf topology the functor $f^\star: \mathcal{F}(S) \to \mathcal{F}(S')_{\text{desc}}$ is an equivalence.*

Example 3.30. *(Descent for elliptic curves.) Consider* Sch/k *together with the fppf topology. Here we choose a cleavage for* $\mathcal{M}_{1,1}$ *and show that* $\mathcal{M}_{1,1} \to \mathrm{Sch}/k$ *(see Example 3.22) is a stack in the fppf topology, hence also in the étale topology. By Lemma 3.27 we know that it is enough to prove that* $f^\star\colon \mathcal{F}(S) \to \mathcal{F}(S')_{\mathrm{desc}}$ *is an equivalence for any covering* $f\colon S' \to S$ *consisting of a single morphism.*

We show that for any covering $f\colon S' \to S$ *the functor* $f^\star\colon \mathcal{M}_{1,1}(S) \to \mathcal{M}_{1,1}(S')_{\mathrm{desc}}$ *is fully faithful. This is a consequence of the result for descent for sheaves on a site stated in Theorem 3.28. Let* X/S *and* X'/S *be in* $\mathcal{M}_{1,1}(S)$*, and let* $f\colon S' \to S$ *be an fppf cover. Suppose that* $\alpha\colon f^\star(X/S) \to f^\star(X'/S)$ *is a morphism in* $\mathcal{F}(S')$ *such that* $p_1^\star\alpha = p_2^\star\alpha$*, where as usual we denote by* p_1 *and* p_2 *the projections* $S' \times_S S' \rightrightarrows S$*. Then* α *is induced by a unique morphism* $h\colon X/S \to X'/S$ *in* $\mathcal{F}(S)$*: to see this, denote by* $h_X\colon {Sch/_S}^{\mathrm{op}} \to \mathrm{Set}$ *the sheaf represented by a scheme* X *over* S *and note that the morphism* α *amounts to giving an isomorphism* $((h_{X\times_S S'}), \phi) \to h_{X'\times_S S'}, \phi')$*, in* $\mathrm{Sh}(S' \to S)_{\mathrm{desc}}$*, where* ϕ*, resp.* ϕ'*, is the canonical covering datum. By Theorem 3.28 this corresponds to a unique isomorphism* $h_X \to h_{X'}$*, which in turn correponds to a unique isomorphism* $X \to X'$ *by the Yoneda lemma. This therefore gives a unique morphism* $X/S \to X'/S$ *in* $\mathcal{F}(S)$*.*

Now, we prove that the functor $f^\star$ *is essentially surjective. To do this we use descent for polarized schemes. First note that every elliptic curve* X/S *has a canonical ample invertible sheaf* $L_{X/S}$ *(see, e.g., [Ols16, 13.1.4]). There is therefore a morphism* $\mathcal{M}_{1,1} \to \mathrm{Pol}$ *of fibered categories given by sending an elliptic curve* $(\pi\colon X \to S, s\colon S \to X)$ *to the pair* $(\pi\colon X \to S, L_{X/S})$*. One then gets the following commutative diagram*

$$\begin{array}{ccc} \mathcal{M}_{1,1}(S) & \xrightarrow{f^\star} & \mathcal{M}_{1,1}(S')_{\mathrm{desc}} \\ \downarrow & & \downarrow \\ \mathrm{Pol}(S) & \xrightarrow{f^\star_{\mathrm{Pol}}} & \mathrm{Pol}(S')_{\mathrm{desc}} \end{array}$$

where $f^\star_{\mathrm{Pol}}$ *is an equivalence, and therefore* $f^\star$ *is essentially surjective.*

4 Algebraic spaces and stacks

At the beginning of Section 3 we motivated the concept of stack – and the related concepts of Grothendieck topology, fibered categories and descent – by explaining that we need a more general notion of sheaf $\mathrm{Sch}/S \to \mathrm{Set}$, since schemes are sheaves $\mathrm{Sch}/S \to \mathrm{Set}$ that are representable. In this sense, a stack satisfies descent conditions which generalise the gluing conditions which are valid for a scheme. On the other hand, the reason we really can work with schemes is that every schemes has a cover by affine schemes. Algebraic stacks are then, roughly, stacks that have a smooth cover by a scheme. When the cover is not only smooth but étale, the algebraic stack is called Deligne-Mumford stack.

We have seen in the previous sections that while the moduli functor $\mathrm{M}_{1,1}$ for elliptic curves is not a sheaf in the Zariski topology, and thus neither in the étale of fppf topology, the category fibered in groupoids $\mathcal{M}_{1,1}$ is a stack in the fppf topology, and thus also in all coarser topologies. In this section we will see that the stack of elliptic curves is a Deligne-Mumford stack, and therefore we have a very nice geometric object encoding isomorphism classes of elliptic curves.

The definition of algebraic stack requires the notion of algebraic spaces, which are sheaves on the étale site of the category of schemes satisfying properties similar to the properties needed to construct a scheme from affine schemes.

4.1 Algebraic spaces

Algebraic spaces are the generalisation of the concept of schemes from the Zariski topology to the étale topology. There are mainly two ways in which one can define an algebraic space, which are derived by the two ways in which one can construe a scheme: as a quotient of an affine scheme by an étale equivalence relation, or as a sheaf in the Zariski topology satisfying certain properties[4]. One therefore defines an algebraic space as a quotient of a scheme by an étale equivalence relation, or as a sheaf in the étale topology that satisfies properties analogous to those that one needs of a Zariski sheaf to be a scheme. The latter definition is often easier to work with, since it does not depend on a particular quotient presentation of the algebraic space. Here we will therefore present the definition of an algebraic space as a sheaf satisfying some properties.

In the following let S be a scheme, and consider the category Sch/S together with the étale topology.

Definition 4.1. *Let F' and F be sheaves on Sch/S, and $f\colon F' \to F$ be a natural transformation. We say that f* is representable *if for any scheme T over S and any natural transformation $h_T \to F$, the fiber product $F' \times_F h_T$ is representable.*

Any morphism between representable sheaves is representable. On the other hand, if we have a representable sheaf F' and a sheaf F, then we could ask for conditions on the sheaf F so that any morphism between the sheaves is representable. In a certain sense, this is the best we can hope for if F is not representable. It turns out that a sufficient condition for this is that the diagonal morphism of F is representable. Recall that the diagonal morphism $\Delta\colon F \to F \times_{h_S} F$ is the unique morphism $F \to F \times_{h_S} F$ whose composition with the projections is the identity on F. In fact we have the following stronger result:

Proposition 4.2. *Let F be a sheaf on Sch/S. The diagonal morphism $\Delta\colon F \to F \times_{h_S} F$ is representable if and only if for any representable sheaf F' any morphism $F' \to F$ is representable.*

Proof. See [Sta14, Tag 025W]. (Note also that the statement is still valid if instead of sheaves one considers presheaves and the fppf topology instead of the étale topology, this is the version of the statement that is proven in [Sta14].) $\square$

Given a representable morphism of sheaves $f\colon F' \to F$ and a property $(\star)$ of morphisms of schemes we say that f has property $(\star)$ if for any representable sheaf X the

[4]For a discussion of how schemes are Zariski sheaves satisfying some properties we refer the reader to [Ols16, 1.4] or [EH00, VI.2.1].

morphism $F' \times_F X \to X$ has property $(\star)$. We are now ready to give the following definition:

Definition 4.3. *An* algebraic space over S *is a sheaf* $X\colon \mathrm{Sch}/S^{\mathrm{op}} \to \mathrm{Set}$ *such that*

(i) The diagonal $\Delta\colon X \to X \times_S X$ *is representable.*

(ii) There exists a scheme U *over* S *and a surjective étale morphism* $h_U \to X$.

Note that every scheme X over S is an algebraic space over S: the diagonal is representable since fiber products exist in the category of schemes, and the identity morphism $id_X\colon X \to X$ is surjective and étale.

Remark 4.4. *Knutson [Knu71] and [LMB00] in addition require the diagonal morphism to be quasi-compact, but to other authors it seems to be unnecessary to add this separation property in the definition of algebraic space, see e.g. the discussion in [Sta14, Tag 025X].*

4.2 Algebraic stacks

Roughly, algebraic stacks are stacks on Sch/S that satisfy the same properties needed from a sheaf on Sch/S to be an algebraic space. Similarly as for algebraic spaces, one can define an algebraic stack as a quotient of an algebraic space by an étale equivalence relation, or as a stack satisfying certain properties. Here we will again discuss the second definition, since it does not depend on a particular presentation of the quotient.

Let U be a scheme over S, then the functor $h_U\colon \mathrm{Sch}/S^{\mathrm{op}} \to \mathrm{Set}$ is the same thing as a category fibered in sets over Sch/S (see the discussion at the end of Section 3.2), and thus can be construed as a category fibered in groupoids over Sch/S. If F is a category fibered in groupoids over Sch/S by a morphism $h_U \to F$ we just mean a morphism of categories fibered in groupoids over Sch/S.

The Yoneda lemma plays a fundamental role in the study of schemes and algebraic spaces, and there is an equivalent 2-categorical version of it which we need for stacks. Let $\mathcal{F}$ be a fibered category over C and let $S \in C$. Then C/S is fibered in sets over C (again, see the discussion at the end of Section 3.2). To any morphism $f\colon C/S \to \mathcal{F}$ of fibered categories over C we can assign the object $f(id_S) \in \mathcal{F}(S)$. This induces a functor

$$\phi_S\colon \mathrm{Hom}_{\mathrm{Fib}/C}(C/S, \mathcal{F}) \to \mathcal{F}(S)$$

where $\mathrm{Hom}_{\mathrm{Fib}/C}(C/S, \mathcal{F})$ is the category with objects morphisms $C/S \to \mathcal{F}$ of categories fibered over C and morphisms natural transformations. Then the 2-categorical version of the Yoneda lemma states that ϕ_S is an equivalence of categories. We refer the reader to [FGI+05, 3.6.2] for details.

Analogously as we have done for algebraic spaces, we define representability for morphisms of algebraic stacks:

Definition 4.5. *A morphism of stacks* $f\colon \mathcal{F}' \to \mathcal{F}$ *is* representable *if for any scheme* U *and any morphism* $g\colon h_U \to \mathcal{F}$ *the* (*2-categorical*) *fiber product* $\mathcal{F}' \times_{\mathcal{F}} h_U$ *is an algebraic space.*

Note that if a morphism of stacks $f\colon \mathcal{F}' \to \mathcal{F}$ is representable, then given any algebraic space V and morphism $V \to \mathcal{F}$, the fiber product $\mathcal{F}' \times_{\mathcal{F}} V$ is an algebraic space. Now, similarly as for algebraic spaces, representability of the diagonal $\Delta\colon \mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ implies that for any scheme U any morphism $h_U \to \mathcal{F}$ is representable.

We are now ready to give the definition of algebraic stack.

Definition 4.6. *A stack $\mathcal{F} \to \mathrm{Sch}/S$ is* algebraic *if the following two conditions are satisfied:*

(i) The diagonal morphism $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable

(ii) There is a scheme U and a smooth surjective morphism $\pi\colon U \to \mathcal{F}$.

Theorem 4.7. *The stack $\mathcal{M}_{1,1}$ is a an algebraic stack over* $\mathrm{Sch}/\mathrm{spec}(\mathbb{Z})$*.*

Proof sketch. We have seen in the Examples 3.22 and 3.30 that $\mathcal{M}_{1,1}$ is a stack. For the proof of the representability of the diagonal, we refer the reader to [Ols16, 13.1.5].

Consider the Weierstrass equation

$$w(X,Y,Z) = ZY^2 + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

and let $\Delta \in \mathbb{Z}[a_1, a_2, a_3, a_4, a_5]$ be the discriminant. Let $\mathbb{A}^5_{\mathbb{Z}} = \mathrm{spec}(\mathbb{Z}[x_1, x_2, x_3, x_4, x_5])$ and let $U = \mathrm{spec}(\mathbb{Z}[a_1, a_2, a_3, a_4, a_5, \Delta^{-1}])$ be the complement of the vanishing locus of the discriminant in $\mathbb{A}^5_{\mathbb{Z}}$. Let $X \subset \mathbb{P}^2_U$ be the solution set of the Weierstrass equation, denote by $\pi\colon X \to U$ the projection, and by $s\colon U \to X$ the section given by the point $(0:1:0)$. We thus have an elliptic curve X/U.

Note that for any scheme S, morphisms $h_S \to \mathcal{M}_{1,1}$ can be identified with objects of $\mathcal{M}_{1,1}(S)$ by the 2-categorical version Yoneda lemma and by the remarks at the end of Section 3.2, therefore the datum of an elliptic curve X over U is the same as a morphism $p\colon h_U \to \mathcal{M}_{1,1}$. Now, p is surjective since every elliptic curve over a field can be given by a Weierstrass equation. For one way to prove that the map p is smooth, we refer the reader to [Ols16, 13.1.9].

□

Definition 4.8. *A* Deligne-Mumford stack *is an algebraic stack $\mathcal{F}$ such that there exists a scheme U and an étale surjective morphism $\pi\colon U \to \mathcal{F}$.*

There is a characterization of Deligne-Mumford stacks, which can be found e.g. in [LMB00, Chapter 8]. (See also the discussion in [Ols16, 8.3.4].)

Theorem 4.9. *Let $X\colon \mathrm{Sch}/S \to$ Groupoid be an algebraic stack. Then X is a Deligne-Mumford stack if and only if the diagonal morphism is unramified (i.e. locally of finite presentation and formally unramified).*

Using this characterisation of Deligne-Mumford stacks one can prove the following (see e.g. [Ols16]):

Theorem 4.10. *The stack $\mathcal{M}_{1,1}$ is a Deligne-Mumford stack.*

To conclude these notes, we list a few of the properties of the stack of elliptic curves $\mathcal{M}_{1,1}$ that one can compute via the étale cover $p\colon h_U \to \mathcal{M}_{1,1}$:

1. Quasi-compactness: one can derive that $\mathcal{M}_{1,1}$ is quasi-compact from the fact that U is [Sta14, Tag 04YC].
2. Its Picard group, which is $\mathbb{Z}/12\mathbb{Z}$ (the original computation for fields of characteristic different from 2 and 3 is in [Mum], while the computation for fields of arbitrary characteristic is in [FO07]).
3. Quasi-coherent sheaves on $\mathcal{M}_{1,1}$ can be obtained from quasi-coherent sheaves on U (see [Ols16, 9.2.12]).

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