Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

The Boardman Vogt resolution and tropical moduli spaces

by

Nina Otter

Master Thesis

submitted to

The Department of Mathematics

Supervisors:

Prof. Dr. John Baez (UCR)

Prof. Dr. Giovanni Felder (ETH)

Contents

Acknowledgements

First and foremost I would like to thank Professor John Baez for making this thesis possible, for his enduring patience and support, and for sharing his lucid vision of the big picture, sparing me the agony of mindlessly hacking my way through the rainforest of topological operads and possibly falling victim to the many perils which lurk within. Furthermore I am indebted to Professor Giovanni Felder who kindly assumed the responsibility of supervising the thesis on behalf of the ETH, and for several fruitful meetings in which he took time to listen to my progress, sharing his clear insight and giving precious advice.

I would also like to thank David Speyer who pointed out the relation between tropical geometry and phylogenetic trees to Professor Baez, and Adrian Clough for all the fruitful discussions on various topics in this thesis and for sharing his ideas on mathematical writing in general.

Finally, I would like to thank my family, in particular my mother and sister, who have always been there for me, even if they do not understand what I do, and why I do it.

Zürich, June 2014.

Introduction

The purpose of this thesis is to relate two seemingly disparate topics: the Boardmann Vogt resolution for operads and moduli spaces of tropical curves of genus 0.

Operads are structures encoding algebras. For every classical algebraic structure (e.g. Lie algebras) it is possible to construct an algebra for which the required identities hold only up to homotopy, or homotopies satisfying higher coherence conditions. To obtain an operad encoding such homotopical algebraic structures one needs to replace the operad encoding algebras by a certain resolution, as for example a cofibrant operad together with a weak equivalence to the original operad.

In the Seventies Boardman and Vogt constructed the resolution bearing their name, also known as W-construction, to encode the structure of a commutative monoid living in homotopy theory. The operations of a free operad are in bijection with trees with vertices labelled by the operations of the operad, and the W-construction is obtained by parametrizing the free operad by assigning lengths to the edges of the trees.

More recently, Berger and Moerdijk constructed a cofibrant resolution for operads in general monoidal model categories, which in the case of the monoidal model category Top of compactly generated spaces is the W-construction by Boardman and Vogt. They proved that this construction gives a cofibrant resolution for the Serre model structure on topological spaces.

Moduli spaces in tropical geometry are the object of recent and ongoing research. As in classical geometry, the simplest moduli spaces are those for curves of genus 0. In [**Mik06**] and [**Mik07**] Mikhalkin investigated the moduli space of tropical curves of genus 0 with n marked points and constructed a compactification which is the tropical analogue of the Deligne-Mumford compactification in algebraic geometry. These moduli spaces are trees with internal edges parametrized by positive real numbers or infinity, and leaves having infinite length.

In classical geometry the Deligne-Mumford compactification of the moduli space of n-pointed stable curves of genus g has the structure of an operad, with composition given by the clutching map in which two curves are glued at a nodal singularity [**GK07**] [**Mar06**] [**Get94**]. In tropical geometry a similar clutching map was recently defined in [**ACP12**]. In Section 9 we show that this clutching map endows the compactification of the tropical moduli space with the structure of a symmetric operad and then show that this operad and W(Comm) are isomorphic as topological non-unital operads, if in W(Comm) we forget the terms of arity 1 and 0.

Preliminaries

1. Monoidal categories

Definition 1. Let $\mathcal{C}$ and $\mathcal{V}$ be two categories. The **product** of $\mathcal{C}$ and $\mathcal{V}$ is a category $\mathcal{C} \times \mathcal{V}$ - with objects pairs (C, V) for all objects C of $\mathcal{C}$ and V of $\mathcal{V}$ and morphisms pairs (f, g) for all morphisms f of $\mathcal{C}$ and g of $\mathcal{V}$, where composition is defined componentwise and with identity morphisms $(1_A, 1_B)$ for every object (A, B) - together with two functors

$$\Pi_1 : \mathcal{C} \times \mathcal{V} \to \mathcal{C} : \begin{cases} (A, B) \to A \\ (f, g) \to f \end{cases}$$

and

$$\Pi_2 : \mathcal{C} \times \mathcal{V} \to \mathcal{V} : \begin{cases} (A, B) \to B \\ (f, g) \to g \end{cases} \quad .$$

A **bifunctor** is a functor $F : \mathcal{C} \times \mathcal{V} \to \mathcal{B}$.
Given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{V} \to \mathcal{Z}$, the **product** $F \times G$ of F and G is a functor
$$F \times G : \mathcal{C} \times \mathcal{V} \to \mathcal{B} \times \mathcal{H} : \begin{cases} (C, V) \mapsto (F(C), G(V)) \\ (f, g) \mapsto (F(f), G(g)) \end{cases}$$

Remark 2. Restricting to small categories, the described operation $\times$ is a functor $\mathbf{Cat} \times \mathbf{Cat} \to \mathbf{Cat}$ and the product of two small categories corresponds to the categorical product in $\mathbf{Cat}$.

The above definition can be extended to n categories and n functors, for n a natural number. We will denote the n-fold product of a category $\mathcal{C}$ by $\mathcal{C}^n$.

Definition 3. A **category with a multiplication** is a category $\mathcal{C}$ together with a covariant bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Given functors $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{A} \to \mathcal{C}$, we denote by $\boldsymbol{F \otimes G}$ the composition $\otimes \circ (F \times G)$. Furthermore, we denote the identity functor on a category $\mathcal{C}$ by $\mathbf{1}_{\mathcal{C}}$, or simply by $\mathbf{1}$ if the category is clear from the context.

Definition 4. A **monoidal category** $\mathcal{V}$ consists of the following data:

- a category $\mathcal{V}_0$

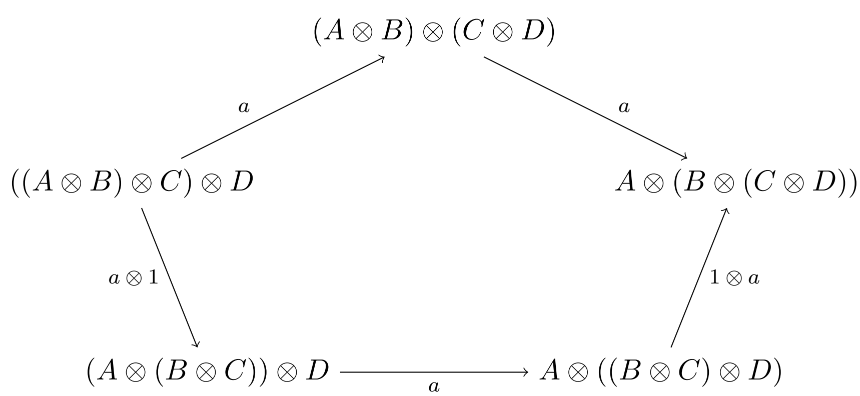
- a bifunctor $\otimes : \mathcal{V}_0 \times \mathcal{V}_0 \to \mathcal{V}_0$
- an object $K \in \mathcal{V}_0$
- natural isomorphisms

$$\begin{aligned} a : (1 \otimes 1) \otimes 1 &\longrightarrow 1 \otimes (1 \otimes 1) \\ \forall A \in ob(\mathcal{C}) : l_A : K \otimes A &\longrightarrow A \\ r_A : A \otimes K &\longrightarrow A \end{aligned}$$

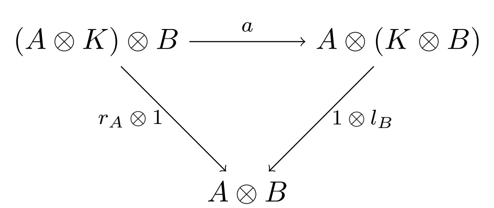
  called **associativity, left unit, right unit constraints**, respectively.

This data satisfies the following axioms:

- **Pentagon for associativity**: the following diagram commutes



- **Triangle for unit**: the following diagram commutes



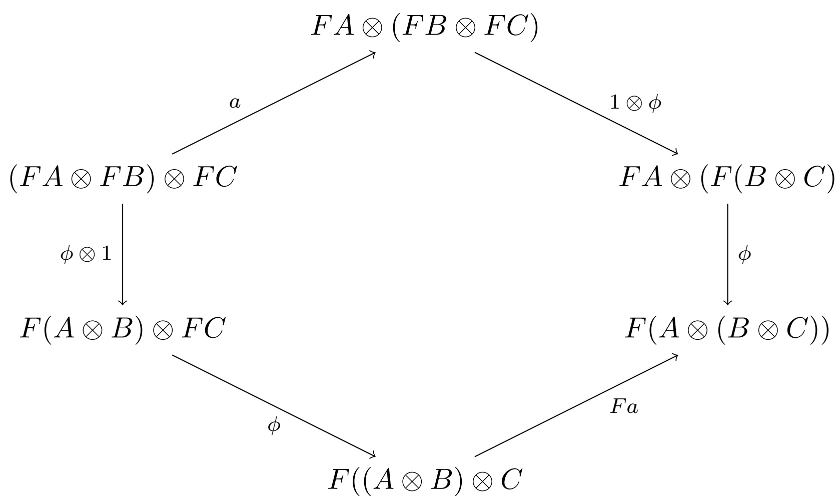
A monoidal category is called **strict** when all the constraints are identity arrows.

Definition 5. Given two monoidal categories $\mathcal{B}$ and $\mathcal{C}$ a **monoidal functor** from $\mathcal{B}$ to $\mathcal{C}$ consists of the following data:

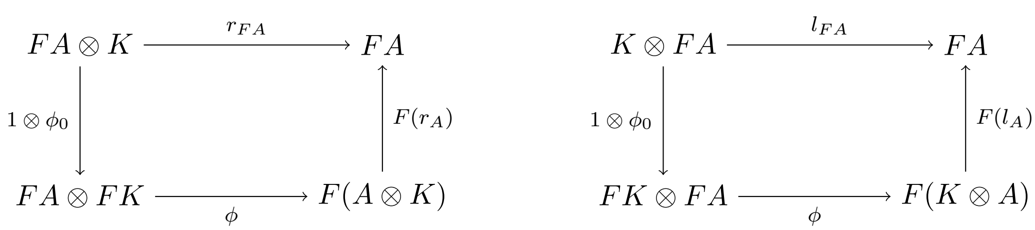
- a functor $F : \mathcal{B}_0 \to \mathcal{C}_0$
- a natural isomorphism $\phi : F1_{\mathcal{B}} \otimes F1_{\mathcal{B}} \to F(1_{\mathcal{B}} \otimes 1_{\mathcal{B}})$
- an isomorphism $\phi_0 : K \to FK$

This data satisfies the commutativity of the following diagrams:

(1.1)



(1.2)



The monoidal functor is called **strict** if ϕ and ϕ_0 are identities.

Definition 6. Given two monoidal functors $\mathcal{B} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{C}$, a **monoidal natural transformation** between them consists of a natural transformation $\theta : F \to G$ which satisfies the commutativity of the two following diagrams:

$$\begin{array}{ccc} FA \otimes FB & \xrightarrow{\phi} & F(A \otimes B) \\ {\scriptstyle \theta_a \otimes \theta_B}\downarrow & & \downarrow{\scriptstyle \theta_{A\otimes B}} \\ GA \otimes GB & \xrightarrow[\phi]{} & G(A \otimes B) \end{array} \qquad \begin{array}{ccc} & K & \\ {\scriptstyle \phi_0}\swarrow & & \searrow{\scriptstyle \phi_0} \\ FK & \xrightarrow[\theta_K]{} & GK \end{array}$$

Definition 7. A **braiding** in a monoidal category $\mathcal{C}$ consists of a family of natural isomorphisms $\{\gamma_{A,B} : A \otimes B \to B \otimes A : A, B \text{ objects in } \mathcal{C}\}$ in $\mathcal{C}$ which satisfy the commutativity of the following diagrams:

$$\begin{array}{ccc} (A \otimes B) \otimes C & \xrightarrow{\gamma_{A,B} \otimes 1} & (B \otimes A) \otimes C \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\ A \otimes (B \otimes C) & & B \otimes (A \otimes C) \\ {\scriptstyle \gamma_{A,B\otimes C}}\downarrow & & \downarrow{\scriptstyle 1 \otimes \gamma_{A,C}} \\ (B \otimes C) \otimes A & \xrightarrow[a]{} & B \otimes (C \otimes A) \end{array} \qquad \begin{array}{ccc} A \otimes (B \otimes C) & \xrightarrow{1 \otimes \gamma_{B,C}} & A \otimes (C \otimes B) \\ {\scriptstyle a^{-1}}\downarrow & & \downarrow{\scriptstyle a^{-1}} \\ (A \otimes B) \otimes C & & (A \otimes C) \otimes B \\ {\scriptstyle \gamma_{A\otimes B,C}}\downarrow & & \downarrow{\scriptstyle \gamma_{A,C} \otimes 1} \\ C \otimes (A \otimes B) & \xrightarrow[a^{-1}]{} & (C \otimes A) \otimes B \end{array}$$

Note that if $\gamma = \{\gamma_{A,B}\}_{A,B\in\mathcal{C}}$ is a braiding, then so is $\gamma' = \{\gamma'_{A,B}\}_{A,B\in\mathcal{C}}$, where $\gamma'_{B,A} = \gamma^{-1}_{A,B}$, since the first diagram for γ gives the second for γ', and conversely.

Remark 8. Note that Maclane includes the commutativity of the diagram

$$\begin{array}{ccc} A \otimes K & \xrightarrow{\gamma_{A,K}} & K \otimes A \\ {\scriptstyle r_A}\searrow & & \swarrow{\scriptstyle l_A} \\ & A & \end{array}$$

in the definition of a braiding. However in [**JS93**] it is shown that the commutativity of this diagram is implied by Definition 7.

Definition 9. A **braided** monoidal category is a monoidal category $\mathcal{C}$ together with a braiding γ. If $\gamma = \gamma'$, then the braided monoidal category is called **symmetric** and the braiding is called **symmetry**.

2. Operads

Let $\mathcal{C} = (\mathcal{C}_0, \otimes, K, a, l, r, s)$ be a symmetric monoidal category. Let **FinSet** denote the category of finite sets and bijections. One of its skeletons is the category $\boldsymbol{\Sigma}$ with objects the finite ordinal numbers $[n]$ for $n \geq 0$. Remark that for $n \neq m$ we have $Hom_{\mathrm{FinSet}}([n],[m]) = \emptyset$ and that the Hom-sets of Σ carry the structure of symmetric groups; we write Σ_n for $Hom_{\mathrm{FinSet}}([n],[n])$. Let $\boldsymbol{\mathcal{C}^{\Sigma^{op}}}$ denote the category of contravariant functors from Σ to $\mathcal{C}$ together with natural transformations between them. Every such functor A gives a collection $\{A(n)\}_{n\in\mathbb{Z}_{\geq 0}}$of objects of $\mathcal{C}$. From now on we drop the brackets for the ordinal numbers.
For each $n \geq 0$ there is a natural action by Σ_n on $A(n)$:

$$\rho : \Sigma_n \to Aut_{\mathcal{C}}(A(n)) : \sigma \mapsto (A(\sigma) : A(n) \to A(n))$$

It is a right action, the functor A being contravariant.

Definition 10. An **operad in $\boldsymbol{\mathcal{C}}$** consists of the following data

- An object of $\mathcal{C}^{\Sigma^{op}}$
- A family of morphisms in $\mathcal{C}$ called **structure morphisms**:

$$\{\gamma_{n;m_1,\ldots,m_n} : \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \ldots \otimes \mathcal{P}(m_n) \to \mathcal{P}(m_1 + \cdots + m_n)\}_{n\geq 1, m_1,\ldots,m_n \geq 0}$$

- A morphism $\eta : K \to \mathcal{P}(1)$ called **unit morphism**

This data is subject to the following axioms:

- Associativity for the structure morphisms: the following diagram commutes for all $n, m_i \geq 1$ and $l_{i,j} \geq 0$ with $1 \leq i \leq n$ and $1 \leq j \leq m_i$:

$$\mathcal{P}(n) \otimes \left(\bigotimes_{i=1}^n \mathcal{P}(m_i)\right) \otimes \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{m_i} \mathcal{P}(l_{i,j})\right)$$

$$\xrightarrow{1\otimes\psi} \mathcal{P}(n) \otimes \left(\bigotimes_{i=1}^n \left(\mathcal{P}(m_i) \otimes \left(\bigotimes_{j=1}^{m_i} \mathcal{P}(l_{i,j})\right)\right)\right) \xrightarrow{1 \otimes \left(\bigotimes_{i=1}^n \gamma_{m_i;l_{i,1},\ldots,l_{i,m_i}}\right)} \mathcal{P}(n) \otimes \left(\bigotimes_{i=1}^n \mathcal{P}\left(\sum_{j=1}^{m_i} l_{i,j}\right)\right) \xrightarrow{\gamma_{n;\sum_{j=1}^{m_1} l_{1,j},\ldots,\sum_{j=1}^{m_n} l_{n,j}}} \mathcal{P}\left(\sum_{i=1}^n \sum_{j=1}^{m_i} l_{i,j}\right)$$

$$\xrightarrow{\gamma_{n;m_1,\ldots,m_n}\otimes 1} \mathcal{P}\left(\sum_{i=1}^n m_i\right) \otimes \left(\bigotimes_{i=1}^n \bigotimes_{j=1}^{m_i} \mathcal{P}(l_{i,j})\right) \xrightarrow{\gamma_{m_1+\cdots+m_n;l_{1,1},\ldots,l_{n,m_n}}} \mathcal{P}\left(\sum_{i=1}^n \sum_{j=1}^{m_i} l_{i,j}\right)$$

where ψ is given by the symmetry and associativity constraint in $\mathcal{C}$.
- Equivariance for the structure morphisms: the following diagrams commute for all $n \geq 1$ and $m_i \geq 0$, for $i = 1, \ldots, n$:

$$
\begin{array}{ccc}
\mathcal{P}(n)\otimes\mathcal{P}(m_1)\otimes\ldots\otimes\mathcal{P}(m_n) & \xrightarrow{\rho(\sigma)\otimes 1} & \mathcal{P}(n)\otimes\mathcal{P}(m_1)\otimes\ldots\otimes\mathcal{P}(m_n) \\
\downarrow{\scriptstyle 1\otimes\bar{\sigma}} & & \\
\mathcal{P}(n)\otimes\mathcal{P}(m_{\sigma^{-1}(1)})\otimes\ldots\otimes\mathcal{P}(m_{\sigma^{-1}(n)}) & & \downarrow{\scriptstyle \gamma_{n;m_1,\ldots,m_n}} \\
\downarrow{\scriptstyle \gamma_{n;m_{\sigma^{-1}(1)},\ldots,m_{\sigma^{-1}(n)}}} & & \\
\mathcal{P}(m_1+\cdots+m_n) & \xrightarrow[\rho(\sigma_{m_1,\ldots,m_n})]{} & \mathcal{P}(m_1+\cdots+m_n)
\end{array}
$$

where $\sigma\in\Sigma_n$ and $\bar{\sigma}$ permutes the factors of the product $\mathcal{P}(m_1)\otimes\ldots\otimes\mathcal{P}(m_n)$ and $\sigma_{m_1,\ldots,m_n}$ denotes the **block permutation**, which for an ordered partition $(m_1,\ldots,m_n)$ of $m=m_1+\cdots+m_n$ sends the ith subinterval of the partition $(m_1,\ldots,m_n)$ to the $\sigma(i)$th subinterval of the partition $\big(m_{\sigma^{-1}(1)},\ldots,m_{\sigma^{-1}(n)}\big)$, and

$$
\begin{array}{ccc}
\mathcal{P}(n)\otimes\mathcal{P}(m_1)\otimes\ldots\otimes\mathcal{P}(m_n) & \xrightarrow{1\otimes\rho(\tau_1)\otimes\cdots\otimes\rho(\tau_n)} & \mathcal{P}(n)\otimes\mathcal{P}(m_1)\otimes\ldots\otimes\mathcal{P}(m_n) \\
\downarrow{\scriptstyle \gamma_{n;m_1,\ldots,m_n}} & & \downarrow{\scriptstyle \gamma_{n;m_1,\ldots,m_n}} \\
\mathcal{P}(m_1+\cdots+m_n) & \xrightarrow[\rho(\widetilde{\tau_1,\ldots,\tau_n})]{} & \mathcal{P}(m_1+\cdots+m_n)
\end{array}
$$

where $\tau_1\in\Sigma_{m_1},\ldots,\tau_n\in\Sigma_{m_n}$ and $(\widetilde{\tau_1,\ldots,\tau_n})$ denotes the image of $(\tau_1,\ldots,\tau_n)$ under the homomorphism $\Sigma_{m_1}\times\cdots\times\Sigma_{m_n}\to\Sigma_{m_1+\cdots+m_n}$.

- Unit: the following diagrams commute for all $n\geq 1$ and all $m\geq 0$:

$$
\begin{array}{ccc}
\mathcal{P}(n)\otimes K^{\otimes n} & \xrightarrow{1\otimes\eta^{\otimes n}} & \mathcal{P}(n)\otimes\mathcal{P}(1)^{\otimes n} \\
{\scriptstyle (r_{\mathcal{P}(n)}\otimes 1)^{\circ n}}\searrow & & \swarrow{\scriptstyle \gamma_{n;1,\ldots,1}} \\
& \mathcal{P}(n) &
\end{array}
$$

$$\begin{array}{ccc} K \otimes \mathcal{P}(m) & \xrightarrow{\eta \otimes 1} & \mathcal{P}(1) \otimes \mathcal{P}(m) \\ & {\scriptstyle l_{\mathcal{P}(m)}} \searrow \quad \swarrow {\scriptstyle \gamma_{1;m}} & \\ & \mathcal{P}(m) & \end{array}$$

Definition 11. If $\mathcal{P}$ and Q are operads in $\mathcal{C}$, a **morphism** $\phi : \mathcal{P} \to Q$ is a family of equivariant morphisms in $\mathcal{C}$

$$\phi_n : \mathcal{P}(n) \to Q(n),\ n \geq 0$$

which preserve the operad structure, that is, such that the following diagrams commute, where γ and η denote the composition and unit of $\mathcal{P}$ and γ' and η' the composition and unit of Q:

$$\begin{array}{ccc} \mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) & \xrightarrow{\gamma_{n;m_1,\ldots,m_n}} & \mathcal{P}(m_1 + \cdots + m_n) \\ \Big\downarrow {\scriptstyle \phi_n \otimes \phi_{m_1} \otimes \cdots \otimes \phi_{m_n}} & & \Big\downarrow {\scriptstyle \phi_{m_1 + \cdots + m_n}} \\ Q(n) \otimes Q(m_1) \otimes \cdots \otimes Q(m_n) & \xrightarrow[\gamma'_{n;m_1,\ldots,m_n}]{} & Q(m_1 + \cdots + m_n) \end{array}$$

$$\begin{array}{ccc} & & \mathcal{P}(1) \\ & {\scriptstyle \eta} \nearrow & \Big\downarrow {\scriptstyle \phi_1} \\ K & & \\ & {\scriptstyle \eta'} \searrow & \\ & & Q(1) \end{array}$$

2.1. Operad algebras.

Definition 12. Let X be an object of $\mathcal{C}$ and $\mathcal{P}$ an operad in $\mathcal{C}$. A **$\mathcal{P}$-algebra structure on X** is a family of morphisms in $\mathcal{C}$

$$\theta_n : \mathcal{P}(n) \otimes X^{\otimes n} \to X,\ n \geq 0$$

that are associative, unital and equivariant in the following sense:

- The following associativity diagram commutes for all $n \geq 1$ and $m_i \geq 0$ for $i = 1, \ldots n$:

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_n) \otimes X^{\otimes m_1+\cdots+m_n} & \xrightarrow{\gamma_{n;m_1,\ldots,m_n} \otimes 1} & \mathcal{P}(m_1+\cdots+m_n) \otimes X^{\otimes m_1+\cdots+m_n} \\
\Big\downarrow{\scriptstyle 1 \otimes m} & & \Big\downarrow{\scriptstyle \theta_{m_1+\cdots+m_n}} \\
 & & X \\
 & & \Big\uparrow{\scriptstyle \theta_n} \\
\mathcal{P}(n) \otimes \mathcal{P}(m_1) \otimes X^{\otimes m_1} \otimes \cdots \otimes \mathcal{P}(m_n) \otimes X^{\otimes m_n} & \xrightarrow[1 \otimes \theta_{m_1} \otimes \cdots \otimes \theta_{m_n}]{} & \mathcal{P}(n) \otimes X^{\otimes n}
\end{array}$$

where, as above, the morphism m is given by the obvious iteration and composition of the braiding and the associativity constraint.

- The following unit diagram commutes:

$$\begin{array}{ccc}
K \otimes X & \xrightarrow{l_X} & X \\
\Big\downarrow{\scriptstyle \eta \otimes 1} & \nearrow{\scriptstyle \theta_1} & \\
\mathcal{P}(1) \otimes X & &
\end{array}$$

- The following equivariance diagram commutes for all $n \geq 0$:

$$\begin{array}{ccc}
\mathcal{P}(n) \otimes X^{\otimes n} & \xrightarrow{\rho(\sigma) \otimes \sigma^{-1}} & \mathcal{P}(n) \otimes X^{\otimes n} \\
{\scriptstyle \theta_n}\searrow & & \swarrow{\scriptstyle \theta_n} \\
 & X^{\otimes n} &
\end{array}$$

where $\sigma \in \Sigma_n$ and $\sigma^{-1}\colon X^{\otimes n} \to X^{\otimes n}$ permutes the factors.

From now on we will denote the action of the symmetric groups by juxtaposition on the right, so instead of $\rho(\sigma)(f)$ we will write $f\sigma$, whenever the former expression makes sense.

3. Trees

This section is taken from joint work with John Baez, which will appear as an article under the title 'Operads and phylogenetic trees' [**BO14**].

Definition 13. For any natural number $n \geq 0$, an ***n*-tree** is a quadruple $T = (V, E, s, t)$ where:

- V is a finite set;
- E is a finite non-empty set whose elements are called **edges**;
- $s\colon E \to V \sqcup \{1, \dots, n\}$ and $t\colon E \to V \sqcup \{0\}$ are maps sending any edge to its **source** and **target**, respectively.

Given $u, v \in V \sqcup \{0, 1, \dots, n\}$, we write $u \xrightarrow{e} v$ if $e \in E$ has $s(e) = u$ and $t(e) = v$.

This data is required to satisfy the following conditions:

- $s\colon E \to V \sqcup \{1, \dots, n\}$ is a bijection;
- there exists exactly one $e \in E$ such that $t(e) = 0$;
- for any $v \in V \sqcup \{1, \dots, n\}$ there exists a **directed edge path** from v to 0: that is, a sequence of edges $e_0, \dots, e_n$ and vertices $v_1, \dots, v_n$ such that

$$v \xrightarrow{e_0} v_1, \; v_1 \xrightarrow{e_1} v_2, \; \dots, \; v_n \xrightarrow{e_n} 0.$$

A **tree** is an n-tree for some $n \geq 0$.

We call 0 the **root**, and call $1, \dots, n$ the **labelled leaves**.

We define the **arity** of a vertex $v \in V$ to be the cardinality of the preimage $t^{-1}(v)$, and call the elements of this preimage the **children** of v. We define a **leaf** to be either a labelled leaf or an **unlabelled leaf**, meaning a vertex of arity zero.

We call an edge **external** if it is has as target or source the root or a leaf, we say that it is **internal** otherwise.

Definition 14. An **isomorphism of *n*-trees** $f\colon (V, E, s, t) \to (V', E', s', t')$ consists of:

- a bijection $f_V\colon V \sqcup \{0, 1, \dots, n\} \to V' \sqcup \{0, 1, \dots, n\}$,
- a bijection $f_E\colon E \to E'$

such that

- f_V is the identity on $\{0, 1, \dots, n\}$,
- $f_V s = s' f_E$,
- $f_V t = t' f_E$.

Definition 15. We call an n-tree with just one vertex a **corolla**.

Definition 16. A **planar** n-tree is an n-tree in which each vertex is equipped with a linear order on the set of its children.

Definition 17. An **isomorphism of planar *n*-trees** is an isomorphism of n-trees $f\colon (V, E, s, t) \to (V', E', s', t')$ that preserves this linear ordering on the children of each vertex.

Definition 18. A **collection** C consists of topological spaces $\{C_n\}_{n\geq 0}$. A **morphism** of collections $f\colon C \to C'$ consists of a continuous map $f_n\colon C_n \to C'_n$ for each $n \geq 0$.

We denote the category of collections by **$\mathbb{N}$-Top**.

Let **Op** be the category consisting of operads and morphisms between them.

Lemma 19. The forgetful functor $U\colon \mathrm{Op} \to \mathbb{N}\text{-Top}$ is **monadic**, i.e. it has a left adjoint F and the category Alg-UF of algebras over the monad $(UF, \eta, 1_U * \epsilon * 1_F)$ induced by the adjunction is equivalent to Op.

PROOF. This is proved in [**BV73**, Defineorem 2.24, Proposition 2.33][1]. □

Definition 20. For any collection C, we define a ***C*-tree** to be an isomorphism class of planar n-trees for which each vertex with k children is labelled by an element of C_k. Here given two planar n-trees (V, E, s, t) and (V', E', s', t') with vertices labelled in this way, we say they are **isomorphic** if there is an isomorphism of planar n-trees $f\colon (V, E, s, t) \to (V', E', s', t')$ such that the labelling of each $v \in V$ equals the labelling of $f_V(v) \in V'$.

Remark that for any morphism of collections $\phi\colon C \to D$ we can define a map $\phi^\star$ from the set of C-trees to the set of D-trees: it assigns to any C-Tree T the D-Tree obtained from T by substituting the label f of any vertex of T by $\phi(f)$.

Definition 21. Consider a planar n-tree $T = (V, E, s, t)$ and a planar m-tree $T' = (V', E', s', t')$. For any $1 \leq i \leq m$ we define the **grafting** (or **composition**) **of *T* on *T'* along *i*** as the $n + m - 1$ tree $T' \circ_i T = (\tilde{V}, \tilde{E}, \tilde{s}, \tilde{t})$ where

- $\tilde{V} = V \sqcup V'$
- $\tilde{E} = \Big(E \setminus \{e_0\}\Big) \sqcup \Big(E' \setminus \{e_i\}\Big) \sqcup \{x\}$, where e_0 is the edge of T with $t(e_0) = 0$ and e_i is the edge of T' such that $s'(e_i) = i$
- $\tilde{s} : \tilde{E} \to \tilde{V} : e \mapsto \begin{cases} s(e), \text{ if } e \in E \text{ and } s(e) \in V \\ s'(e), \text{ if } e \in E' \text{ and } s'(e) \in V' \\ s'(e), \text{ if } e \in E' \text{ and } 1 \leq s'(e) \leq i-1 \\ s(e) + i - 1, \text{ if } e \in E \text{ and } 1 \leq s(e) \leq n \\ s'(e) + n - 1, \text{ if } e \in E' \text{ and } i+1 \leq s'(e) \leq m \\ s(e_0) \text{ if } e = x \end{cases}$
- $\tilde{t} : \tilde{E} \to \tilde{V} : e \mapsto \begin{cases} t(e), \text{ if } e \in E \\ t'(e), \text{ if } e \in E' \\ t(e_i), \text{ if } e = x \end{cases}$

If in T the order of the children of $t(e_i)$ is $e_1 < \cdots < e_{i-1} < e_i < e_{i+1} < \cdots < e_r$, then the order of its children in $T \circ_i T'$ is $e_1 < \cdots < e_{i-1} < x < e_{i+1} < \cdots < e_r$. The order of the children of all other vertices is unchanged.

We say that edge e_0 is **identified** with edge e_i.

[1] This reference was pointed out by Richard Garner.

We can extend composition of trees to isomorphism classes of trees:

Lemma 22. Composition of planar trees is well-defined on isomorphism classes.

PROOF. Let T, T' be two planar n-trees and S, S' two planar m-trees together with isomorphisms $f\colon T \to T'$ and $g\colon S \to S'$. Then f and g induce an isomophism $f \circ_i g\colon T \circ_i S \to T' \circ_i S'$ which is given on vertices by the disjoint union of f_V and g_V and on edges by

$$e \mapsto \begin{cases} f_E(e), \ e \in E(T) \\ g_E(e), \ e \in E(S) \\ x' \in E(T' \circ_i S') \setminus \big(E(T') \sqcup E(S')\big), \text{otherwise.} \end{cases}$$

If f and g respect the planar structure, then so does $f \circ_i g$. □

The action of the symmetric groups on operations of $F(C)$ is given by 'relabelling of leaves'.

Definition 23. Given a planar n-tree $T = (V, E, s, t)$ and a permutation $\sigma \in \Sigma_n$, we define $T\sigma = (V, E, s\sigma, t)$, where $s\sigma\colon E \to V \sqcup \{1, \dots, n\}\colon e \mapsto \begin{cases} s(e), \text{ if } s(e) \in V \\ \sigma^{-1}(s(e)) \text{ otherwise} \end{cases}$

This operation defines an action of the symmetric group Σ_n on the set of planar n-trees. To define the colift of a morphism of collections $C \to U(\mathcal{P})$ along the monomorphism $C \to F(C)$ which is part of the data of the free operad on C, we need to define contractions of edges for an $U(\mathcal{P})$-tree.

Definition 24. Consider a planar n-tree $T = (V, E, s, t)$ and suppose that it has an internal edge e. We can assign to it a planar n-tree $\boldsymbol{T \setminus e} = (V_e, E_e, s_e, t_e)$ in which the edge e is deleted:

- the vertex set of $T \setminus e$ is given by $\Big(V \setminus \{s(e), t(e)\}\Big) \sqcup \{x\}$
- the edge set is given by $E \setminus \{e\}$
- The maps s_e and t_e are defined as follows:

$$s_e\colon E_e \to V_e \sqcup \{1, \dots, n\}\colon e' \mapsto \begin{cases} s(e') \text{ if } s(e') \neq t(e) \\ x, \text{ otherwise} \end{cases}$$

$$t_e\colon E_e \to V_e \sqcup \{0\}\colon e' \mapsto \begin{cases} t(e'), \text{if } t(e') \neq t(e) \text{ and } t(e') \neq s(e) \\ x, \text{ otherwise} \end{cases}$$

The order on the children of x is defined as follows: suppose that $t(e)$ has $k_1 > 0$ children, while $s(e)$ has $k_2 > 0$ children, and that e is the ith smallest children of $t(e)$. The planar order induces order-preserving isomorphisms $\phi_1 \to \mathrm{in}(t(e)) \to [k_1]$ and $\phi_2\colon \mathrm{in}(s(e)) \to [k_2]$, where we denote by $\mathbf{in}(\boldsymbol{v})$ the set of children of a vertex v. Now we define

$$\phi_1 \circ_i \phi_2\colon \mathrm{in}(t(e)) \sqcup \mathrm{in}(s(e)) \setminus \{e\} \to [k_1 + k_2 - 1]\colon y \mapsto \begin{cases} \phi_1(y), \text{ if } y \in \mathrm{in}(t(e)) \text{ and } 1 \leq \phi_1(y) \leq i - 1 \\ \phi_2(y) + i - 1, \text{ if } y \in \mathrm{in}(s(e)) \\ \phi_1(y) + k_1 - 1, \text{ if } y \in \mathrm{in}(t(e)) \text{ and } \phi_1(y) > i \end{cases}$$

This induces a linear order on $\mathrm{in}(x)$.

This operation is well-defined on isomorphism classes of trees, and we can extend it to $U(\mathcal{P})$-trees, for any operad $\mathcal{P}$. To do so we need to use the definition of partial composition for operads:

Definition 25. For all $k \geq 1$ and $l \geq 0$ and for all $f \in \mathcal{P}(k)$ and $g \in \mathcal{P}(l)$ and all $1 \leq i \leq k$ we define

$$f \circ_i g = \gamma_{k;1,\dots,1,l,1,\dots,1}(f, \mathrm{id}_{\mathcal{P}}, \dots, \mathrm{id}_{\mathcal{P}}, g, \mathrm{id}_{\mathcal{P}}, \dots, \mathrm{id}_{\mathcal{P}})$$

with g at the ith position.

It is possible to define operads using this partial composition and this yields a definition equivalent to the definition we have given in Section 2, see for example [**May97**, Definition 12].

Now suppose that the vertices $s(e)$ and $t(e)$ are labelled by the operations $g \in \mathcal{P}(k_2)$ and $f \in \mathcal{P}(k_1)$, respectively; then we assign to the vertex x the label $f \circ_i g$. This assignment yields again an $U(\mathcal{P})$-tree: we have $f \circ_i g \in \mathcal{P}(k_1 + k_2 - 1)$, and by definition x has $k_1 + k_2 - 1$ children. We call this operation **contraction of edge e**.

Reiterating this operation, we can assign to any $U(\mathcal{P})$-tree T with n labelled leaves a unique $U(\mathcal{P})$-tree which is a corolla with n leaves and with the unique vertex labelled by the composition of the labels of the vertices of T: this assignment does not depend on the order in which we delete the inner edges, since the composition in $\mathcal{P}$ is associative.

Definition 26. We denote by $\boldsymbol{\gamma(T)}$ the label of the vertex of the corolla obtained by contracting all internal edges of the tree T.

On the other hand, instead of contracting all internal edges of a tree, we could contract just those of a subtree; by the latter, we mean a subset of edges and vertices of a tree T which inherits from it the structure of a tree.

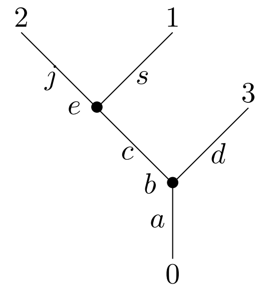
Definition 27. [**Fre**] A **subtree** $S = (V(S), E(S), \mathrm{in}(S), 0_S, s, t)$ of a planar n-tree $T = (V(T), E(T), s, t)$ is given by:

- a non-empty set of vertices $V(S) \subset V(T)$
- a set of edges $E(S) \subset E(T)$
- a set $\mathrm{in}(S) \subset V(T) \sqcup \{1, \dots, n\}$, such that $\mathrm{in}(S) \cap V(S) = \emptyset$
- an element $0_S \in V(T)$ such that $\{0_S\} \cap V(S) = \emptyset$ and such that there is a unique edge e_0 in $E(S)$ with $t(e_0) = 0_S$
- the restrictions of s and t to $E(S)$.

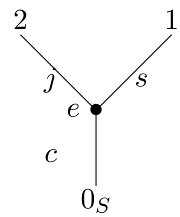
This data satisfies the following requirement: an edge e is in $E(S)$ iff $t(e) \in V(S) \sqcup \{0_S\}$ iff $s(e) \in V(S) \sqcup \mathrm{in}(S)$.

The last requirement in the definition insures that in a subtree S there is a unique directed edge path from any vertex to 0_S, and also that if a vertex is in $V(S)$ then all its children and the edge with the vertex as its source are in $E(S)$. Furthermore, a subtree is completely determined by its set of vertices or its set of edges, see [**Fre**, A 1.6].

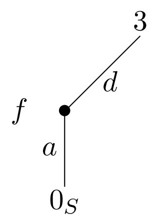
For example: given the 3-tree



this is a subtree:



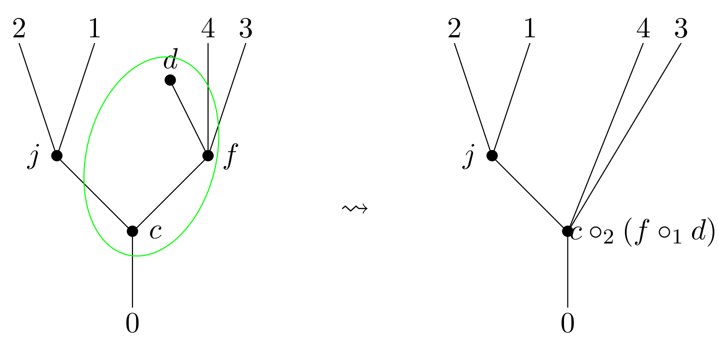
while this is not:



The definition of subtrees can be extended in the obvious way to C-trees. We call an edge $e \in E(S)$ **internal** if $t(e) \in V(S)$ and $s(e) \in V(S)$, **external** otherwise.

Contracting the internal edges of a subtree S of an $U(\mathcal{P})$-tree we obtain again an $U(\mathcal{P})$-tree [**Fre**] and we call this operation **contraction of the subtree *S***.

For example, we can contract the subtree given by the vertices encircled by the green ellipse of the following tree:



Lemma 28. Let C be a collection. There is an operad $C\mathrm{Tree}$ such that:

(3.1) $C\mathrm{Tree}(n)$ is the set of C-trees with n labelled leaves;

(3.2) there is a unique isomorphism of operads

$$\psi\colon F(C) \xrightarrow{\sim} C\mathrm{Tree}$$

that for each $n \geq 0$ sends $f \in C_n$ to the isomorphism class of the corolla with its n labelled leaves ordered so that $1 < \cdots < n$, and with its vertex labelled by f.

Proof. First, we construct an operad $C\mathrm{Tree}$ such that $C\mathrm{Tree}(n)$ is the set of C-trees with n labelled leaves. Composition is given by grafting of trees, with unit given by the isomorphism class of the tree with empty set of vertices. The assignment $C\mathrm{Tree}(n) \times \Sigma_n \to C\mathrm{Tree}(n)\colon (T, \sigma) \mapsto T\sigma$ is well-defined and defines an action of Σ_n on $C\mathrm{Tree}(n)$, and composition of trees is equivariant with respect to this action.

Second, we show that there is an isomorphism ψ with property (2). We define a morphism of collections

$$\iota\colon C \to U(C\mathrm{Tree})$$

which sends an element $f \in C_n$ to the isomorphism class of the corolla with its n labelled leaves ordered so that $1 < \cdots < n$, and with its vertex labelled by f. Next we show that the pair $(C\mathrm{Tree}, \iota)$ satisfies the universal property of the free operad on C.

Let Q be an operad and let $\phi\colon C \to U(Q)$ be a morphism of collections. We define a morphism $\overline{\phi}\colon C\mathrm{Tree} \to Q$ as follows:

- $\overline{\phi}(1)(id_{C\mathrm{Tree}}) = id_Q$
- $\overline{\phi}(n)(\iota(f)) = \phi(f)$ for all $f \in C_n$
- $\overline{\phi}(n)(T) = \gamma(\phi^\star(T))$ for all trees $T \in C\mathrm{Tree}(n)$ with leaves labelled by $1 < \cdots < n$
- $\overline{\phi}(n)(T \cdot \sigma) = \overline{\phi}(T)\sigma$ for all $T \in C\mathrm{Tree}(n)$ and for all $\sigma \in \Sigma_n$.

To see that this assignment preserves the operadic composition, remark first that for any pair of trees $T \in C\mathrm{Tree}(n)$ and $T' \in C\mathrm{Tree}(m)$ with unpermuted leaves we have $\phi^\star(T \circ_i T') = \phi^\star(T) \circ_i \phi^\star(T')$. So it remains to show that $\gamma\left(\phi^\star(T) \circ_i \phi^\star(T')\right) = \gamma\left(\phi^\star(T)\right) \circ_i \gamma\left(\phi^\star(T')\right)$. First, we contract the subtrees S and S' of $\phi^\star(T \circ_i T')$ which are given by the sets of vertices of T and T', respectively. We thus obtain a tree with exactly one internal edge e, namely the edge arising from the grafting of T' to T, while the vertex $t(e)$ is labelled by $\gamma(\phi^\star(T))$ and the vertex $s(e)$ by $\gamma(\phi^\star(T'))$. Furthermore, the edge e is the ith smallest children of $t(e)$. To see this, recall that in $T \circ_i T'$ there is exactly one directed edge path from every leaf to the root, and during the contraction process all internal edges on these paths but e are contracted, so there are exactly $i - 1$ children of $t(e)$ which are external edges and whose leaves are smaller than the smallest leaf whose directed edge path to the root passes through the edge e. Now, if we contract the edge e we obtain $\gamma(\phi^\star(T)) \circ_i \gamma(\phi^\star(T'))$. On the other hand, since the order in which we contract the edges is immaterial, if we contract the edge e we also obtain $\gamma(\phi^\star(T) \circ_i \phi^\star(T'))$.

We thus have defined a morphism of operads $\overline{\phi}\colon C\text{Tree} \to Q$ which colifts ϕ along ι, and by construction $\overline{\phi}$ is the unique morphism with this property. Hence there is a unique isomorphism

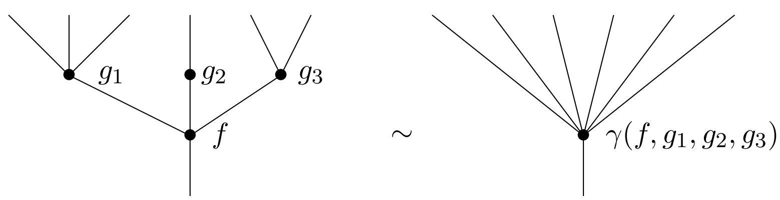
$$\psi\colon F(C) \xrightarrow{\sim} C\text{Tree}$$

sending the image of $f \in C_n$ in $F(C)$ to $\iota(f)$. □

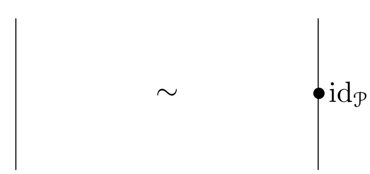
Remark that we can also define a **permutation on subtrees** of a tree consisting of a single vertex, by permuting the children of the vertex. More precisely, for a subtree S consisting of a single vertex, the linear order on $\text{in}(S)$ gives an order-preserving isomorphism $f\colon \text{in}(S) \to [k]$ for some $k \geq 0$. Then we define the permutation of $\text{in}(S)$ by $\sigma \in \Sigma_k$ to be the composition $\sigma^{-1} \circ f$.

Lemma 29. Let $\mathcal{P}$ be an operad. Then $\epsilon_{\mathcal{P}}$ maps two $\mathcal{P}$-trees to the same operation of $\mathcal{P}$ if and only if we can go from one $\mathcal{P}$-tree to the other by a finite sequence of the following moves:

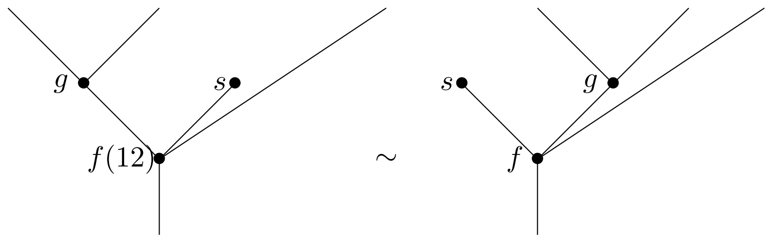
(3.1) Given any $\mathcal{P}$-tree, we can replace any subtree consisting of a vertex together with its children and their source vertices by its contraction.
For example:



(3.2) For any $\mathcal{P}$-tree we can replace any edge by a corolla with one vertex labelled by the identity $\text{id}_{\mathcal{P}} \in \mathcal{P}(1)$:



(3.3) For any $\mathcal{P}$-tree we can replace any subtree S given by exactly one vertex v labelled by $f\sigma$, where $\sigma \in \Sigma_k$ and $f \in \mathcal{P}(k)$, by the subtree obtained from S by permuting $\text{in}(S)$ by σ^{-1} and substituting the label of v by f.
For example:



Proof. First we fix some notation for $U(\mathcal{P})$-trees:

- $\boldsymbol{s_{f,g_1,\dots,g_k}}$ is the subtree of a tree which is given by a vertex labelled by f together with its children and their sources, which are labelled in order by $g_1,\dots,g_k$.
- $\boldsymbol{s_f}$ is the subtree of a tree which is given by exactly one vertex labelled by f
- $\boldsymbol{s_f\sigma}$ is the subtree s_f permuted by $\sigma\in\Sigma_n$, where $f\in\mathcal{P}(n)$
- $\boldsymbol{t_{f,g_1,\dots,g_k}}$ is a tree with leaves labelled by $1<\dots<n$ and with exactly k internal edges which are all the children of the same vertex, so that this vertex is labelled by $f\in\mathcal{P}(k)$, while the other vertices are labelled in order by $g_1,\dots,g_k$.
- $\boldsymbol{c_f}$ is the corolla with n leaves labelled by $1<\dots<n$ and with the unique vertex labelled by $f\in\mathcal{P}(n)$

By Lemma 19 we know that $(\mathcal{P},\epsilon_{\mathcal{P}})$ is the coequalizer of the following diagram:

$$FUFU(\mathcal{P}) \underset{FU(\epsilon_{\mathcal{P}})}{\overset{\epsilon_{FU(\mathcal{P})}}{\rightrightarrows}} FU(\mathcal{P})$$

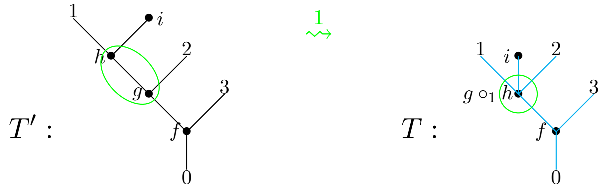
and also that the forgetful functor creates coequalizers for those parallel pairs (f,g) in Op so that (Uf,Ug) has a split coequalizer in $\mathbb{N}$-Top, so we can give the description of $\mathcal{P}$ as a coequalizer in $\mathbb{N}$-Top, which again is given entrywise in Top.

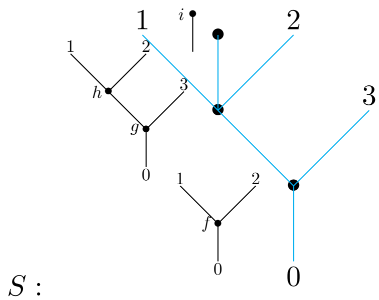
In the following we say that two $\mathcal{P}$-trees $T,T'\in FU(\mathcal{P})(n)$ are **equivalent** iff (T,T') is an element of the smallest equivalence relation generated by $\{(\epsilon_{FU\mathcal{P}}(S), FU\epsilon_{\mathcal{P}}(S)) \mid S\in FUFU\mathcal{P}(n)\}$.

If two trees in $FU(\mathcal{P})(n)$ differ by a finite sequence of moves 3.1, 3.2, 3.3, then there are n-trees $T_1,\dots T_n$ such that $T_1=T$, $T_n=T'$ and T_i and T_{i+1} differ by exactly one of the moves 3.1, 3.2, 3.3, for $i=1,\dots,n-1$. So it is enough to consider the case in which T and T' differ by exactly one of the moves 3.1, 3.2 or 3.3.

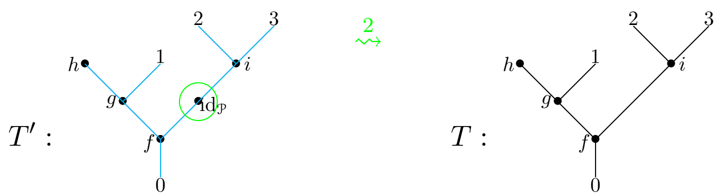
First suppose that T and T' are $\mathcal{P}$-trees in $FU(\mathcal{P})(n)$ with unpermuted leaves and which differ by exactly one of the moves 3.1 or 3.2. Then there exists $S\in FUFU(\mathcal{P})(n)$ such that $FU\epsilon_{\mathcal{P}}(S)=T$ and $\epsilon_{FU\mathcal{P}}(S)=T'$:

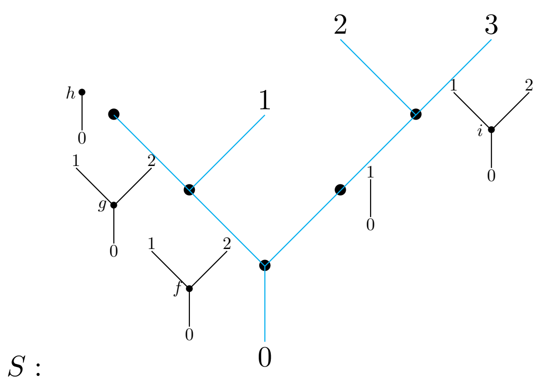
- Suppose $T'\in FU(\mathcal{P})(n)$ has a subtree $s_{f,g_1,\dots,g_k}$ and that T is obtained from T' by applying move 3.1 to $s_{f,g_1,\dots,g_k}$, where we denote the vertex corresponding to the contracted subtree by v'. Then S is obtained from T by substituting the label f of every vertex $v\neq v'$ with the corolla c_f, and the label of v' by $t_{f,g_1,\dots,g_k}$. For example:





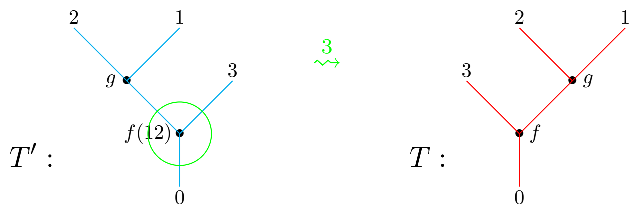
- Suppose that T' contains a subtree $s_{\mathrm{id}_{\mathcal{P}}}$ and that T is obtained from T' by application of move 3.2 to $s_{\mathrm{id}_{\mathcal{P}}}$, where we denote the vertex labelled by $\mathrm{id}_{\mathcal{P}}$ by v'. Then S is obtained from T' by substituting the label f of every vertex $v \neq v'$ with the corolla c_f, and the label of v' by the trivial tree. For example:

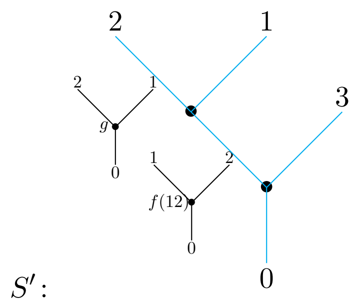


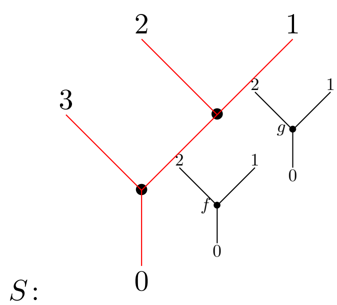


Now suppose that T and T' do not have necessarily unpermutated leaves and that T' contains a subtree $s_{f\sigma}$ while T is obtained from T' by applying move 3.3 to $s_{f\sigma}$. Then there exist S and S' in $FUFU(\mathcal{P})(n)$ such that $T = \epsilon_{FU(\mathcal{P})}(S)$ and $T' = \epsilon_{FU(\mathcal{P})}(S')$, and

$FU\epsilon_{\mathcal{P}}(S) = FU\epsilon_{\mathcal{P}}(S')$: S is obtained from T by substituting every label by the corolla with vertex labelled by that label, while S' is obtained in the same way from T'. For example:





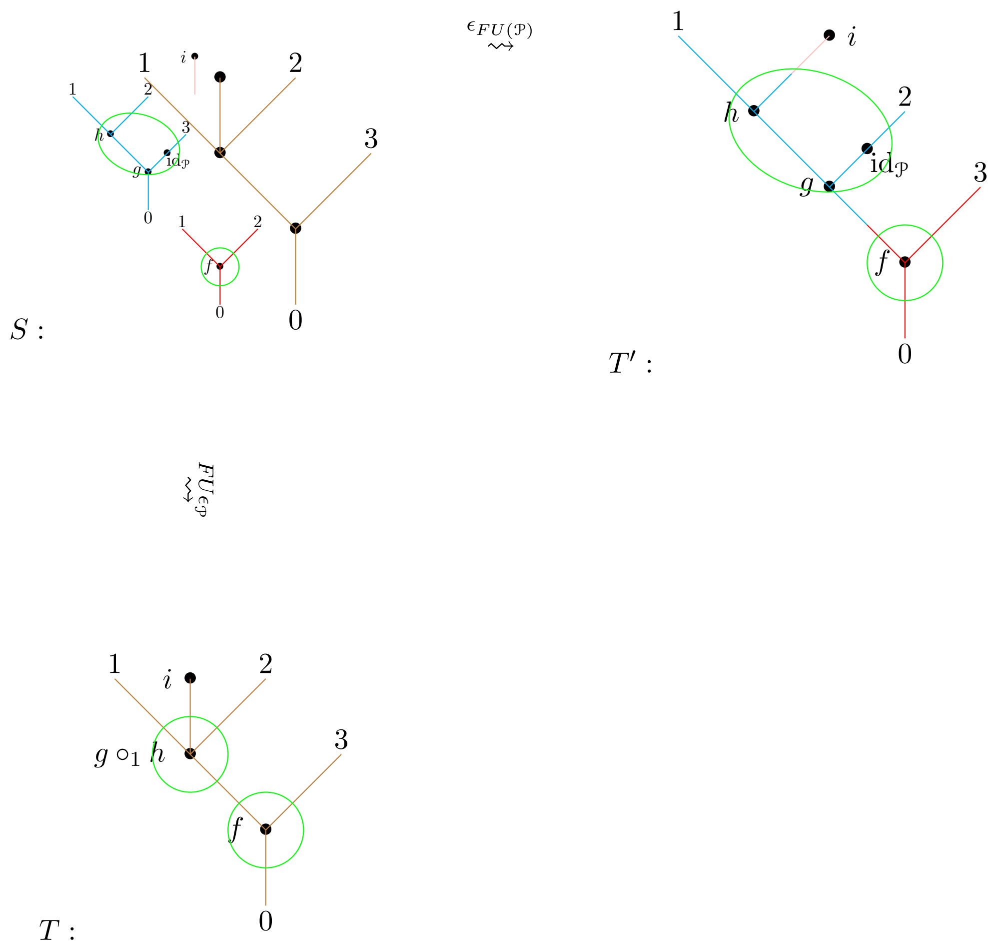


So we have seen that if T and T' are two trees with any labelling of the leaves which differ by exactly one of the moves 3.1, 3.2, or 3.3, then they are equivalent.

Conversely, we have to show that for any tree $S \in FUFU(\mathcal{P})(n)$ we can go from $\epsilon_{FU\mathcal{P}}(S)$ to $FU\epsilon_{\mathcal{P}}(S)$ with a finite sequence of moves $3.1, 3.2, 3.3$. If this is true, then it is for all elements of the smallest equivalence relation generated by the pairs $(\epsilon_{FU\mathcal{P}}(S), FU\epsilon_{\mathcal{P}}(S))$. We set $\epsilon_{FU\mathcal{P}}(S) = T'$ and $FU\epsilon_{\mathcal{P}}(S) = T$.

If all vertices of S are labelled by a corolla with unpermuted leaves, then $T = T'$. Now suppose that a vertex of S is labelled by a tree t different from the trivial tree and with unpermuted leaves, while all the remaining vertices are labelled by unpermuted corollas. To this label corresponds in T a subtree $s_{\gamma(t)}$, while in T' it corresponds to a subtree with planar structure and vertex set given by t, hence T and T' are related by a sequence of finite moves of type 3.1 or 3.2.

For example:



If in addition some vertex of S is labelled by a trivial tree, then we can go from T to T' by an additional finite sequence of moves of type 3.2. Finally, if in addition some tree labelling S has permuted leaves, then we can go from T to T' by an additional finite sequence of moves of type 3.3.

□

Lemma 30. Let $\mathcal{P}$ and $\mathcal{P}'$ be operads. Operations in $F(U(\mathcal{P}) + (U(\mathcal{P}'))$ may be identified with $U(\mathcal{P}) + U(\mathcal{P}')$-trees. Two $U(\mathcal{P}) + U(\mathcal{P}')$-trees map to the same operation of $\mathcal{P} + \mathcal{P}'$ via the operad homomorphism

$$\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'} \colon F(U(\mathcal{P}) + U(\mathcal{P}')) \to \mathcal{P} + \mathcal{P}'$$

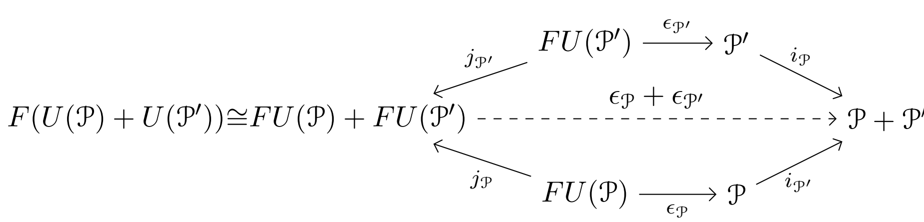
if and only if we can go from one to the other by a finite sequence of the following moves:

(3.1) For any $U(\mathcal{P})+U(\mathcal{P}')$-tree, we can replace any $U(\mathcal{P})$-subtree by its contraction, where a ***U*(𝒫)-subtree** is a subtree having vertices labelled only by operations of $U(\mathcal{P})$.
(3.2) For any $U(\mathcal{P})+U(\mathcal{P}')$-tree, we can replace any edge by a corolla with one vertex labelled by the identity $\mathrm{id}_{\mathcal{P}} \in \mathcal{P}_1$.
(3.3) For any $U(\mathcal{P})+U(\mathcal{P}')$-tree, we can replace any subtree $s_{f\sigma}$, where $\sigma \in S_k$ and $f \in \mathcal{P}_k$, by $s_f\sigma$.
(3.4) The same as (1) with $\mathcal{P}'$ instead of $\mathcal{P}$.
(3.5) The same as (2) with $\mathcal{P}'$ instead of $\mathcal{P}$.
(3.6) The same as (3) with $\mathcal{P}'$ instead of $\mathcal{P}$.

Proof. We begin by giving a description of the morphism

$$\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'} \colon FU(\mathcal{P}) + FU(\mathcal{P}') \to \mathcal{P} + \mathcal{P}'$$

and will then refer to the composite morphism $F(U(\mathcal{P})+U(\mathcal{P}')) \xrightarrow{\sim} FU(\mathcal{P}) + FU(\mathcal{P}') \to \mathcal{P} + \mathcal{P}'$ by using the same symbol. The morphism $\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}$ is induced by the universal property of the coproduct $\mathcal{P} + \mathcal{P}'$; namely, it is the unique morphism $FU(\mathcal{P}) + FU(\mathcal{P}') \to \mathcal{P} + \mathcal{P}'$ making the following diagram commute, where the morphisms $j_{\mathcal{P}}, j_{\mathcal{P}'}$ and $i_{\mathcal{P}}, i_{\mathcal{P}'}$ are part of the data of the coproducts:

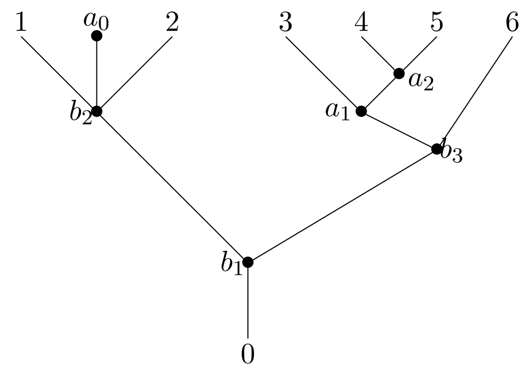


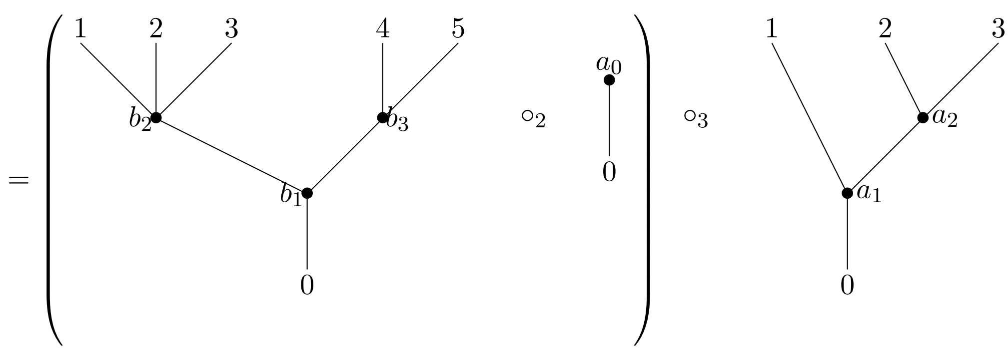
We know from Lemma 29 that operations in $\mathcal{P}+\mathcal{P}'$ are equivalence classes of $U(\mathcal{P})+U(\mathcal{P}')$-trees, while operations in $F(U(\mathcal{P})+U(\mathcal{P}'))$ are $U(\mathcal{P})+U(\mathcal{P}')$-trees by Lemma 28. We have for all $n \geq 0$:

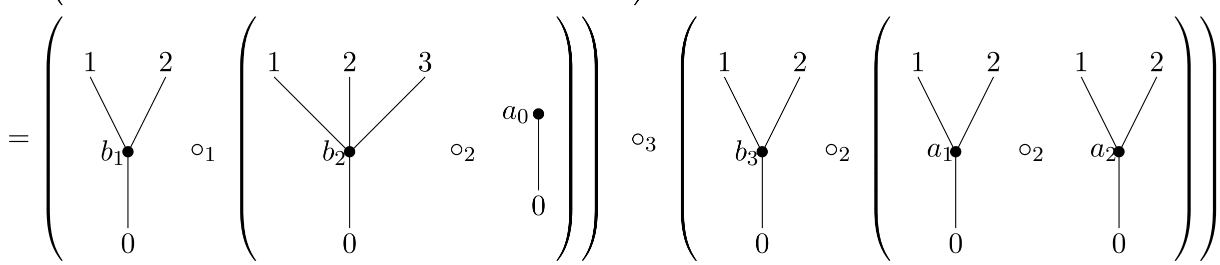
$$i_{\mathcal{P}n} \colon \mathcal{P}_n \to (\mathcal{P}+\mathcal{P}')_n \colon f \mapsto c_f$$

while $j_{\mathcal{P}n} \colon FU(\mathcal{P})_n \to F(U(\mathcal{P})+U(\mathcal{P}'))_n$ is just the inclusion.

Given a $U(\mathcal{P})$-tree T, we define $\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}(T) = i_{\mathcal{P}}(\epsilon_{\mathcal{P}}(T))$ and analogously for $\mathcal{P}'$. Given a $U(\mathcal{P})+U(\mathcal{P}')$-tree T, if its leaves are permuted by σ, then $\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}(T) = \epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}(\widetilde{T})\sigma$, with $\widetilde{T}\sigma = T$, so we may assume that the leaves of T are unpermuted. We can decompose any $U(\mathcal{P})+U(\mathcal{P}')$-tree in $U(\mathcal{P})$- and $U(\mathcal{P}')$-trees. Note that this decomposition is not necessarily unique. For example consider the following decompositions, where we assume that a_i are operations of $\mathcal{P}$ and b_i of $\mathcal{P}'$:







We write

$$T = S_1 \circ_{j_1} \Big(S_2 \circ_{j_2} \big(\dots (S_k \circ_{j_k} S_{k+1}) \dots \big) \Big)$$

for such a decomposition.

It is possible to define a partial order on the set of decompositions of a tree: a decomposition D is smaller then a decomposition D' if by substituting none or a finite number of trees in D by their decomposition we obtain D'. With this partial order the set of decompositions of a tree is a bounded lattice, hence for every tree there is a maximum and a minimum decomposition. In the tree above the first decomposition is the minimum decomposition, while the second one is the maximum decomposition.

We then define

$$\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}(T) = i_1\epsilon_1(S_1) \circ_{j_1} \Big(i_2\epsilon_2(S_2) \circ_{j_2} \big(\dots \big(i_k\epsilon_k(S_k) \circ_{j_k} i_{k+1}\epsilon_{k+1}(S_{k+1})\big) \dots\big)\Big)$$

where we write $i_p\epsilon_p$ instead of $i_p \circ \epsilon_p$ and $i_p\epsilon_p = \begin{cases} i_{\mathcal{P}}\epsilon_{\mathcal{P}} \text{ if } S_p \text{ is a } U(\mathcal{P})\text{-tree} \\ i_{\mathcal{P}'}\epsilon_{\mathcal{P}'} \text{ otherwise} \end{cases}$

for all $1 \leq p \leq k+1$.

Since $i_{\mathcal{P}}\epsilon_{\mathcal{P}}$ is a morphism of operads, we have for any pair of $U(\mathcal{P})$-trees T and T'

$$i_{\mathcal{P}}\epsilon_{\mathcal{P}}(T \circ_i T') = i_{\mathcal{P}}\epsilon_{\mathcal{P}}(T) \circ_i i_{\mathcal{P}}\epsilon_{\mathcal{P}}(T')$$

and similarly for $\mathcal{P}'$, hence the definition of $\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}$ does not depend on the particular decomposition of a tree, and $\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}$ preserves the operadic composition.

Now, remark that $\epsilon_{\mathcal{P}} + \epsilon_{\mathcal{P}'}$ sends two trees T and T' to the same equivalence class in $\mathcal{P} + \mathcal{P}'$ iff $i_1\epsilon_1(S_1) \circ_{j_1} \dots \circ_{j_{k-1}} i_k\epsilon_k(S_k) = i_1\epsilon_1(S'_1) \circ_{j'_1} \dots \circ_{j'_{k'-1}} i_{k'}\epsilon_{k'}(S'_{k'})$.

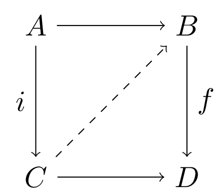
If we take the minimum decomposition of T and T', we necessarily have $k = k'$ and $i_p\epsilon_p(S_p) = i_p\epsilon_p(S'_p)$ for all $1 \leq p \leq k$. This is equivalent to $\epsilon_p(S_p) = \epsilon_p(S'_p)$, since i_p is injective. By Lemma 29 we know that this is the case if and only if we can go from S_p to $S_{p'}$ with a finite sequence of moves 1, 2, 3. Since any $U(\mathcal{P})$-tree in a decomposition of a $U(\mathcal{P}) + U(\mathcal{P}')$-tree determines a unique $U(\mathcal{P})$-subtree (and analogously for $\mathcal{P}'$), we obtain the desired result. □

Operads and homotopy theory

4. Monoidal model categories

We briefly recall the definition of a model category and monoidal model category. Standard references for this are [**Qui67**], [**Hir09**], [**Hov07**], and [**SS98**] for the definition of monoidal model category.

Definition 31. A morphism f in a category $\mathcal{C}$ is said to have the **right lifting property** or shortly **RLP** with respect to a class of morphisms I of $\mathcal{C}$ if the dashed arrows exists in any diagram of the form



with $i\colon B \to D$ a morphism in I. Similarly, a morphism i is said to have the **left lifting property** or **LLP** with respect to a class of morphisms F in $\mathcal{C}$, if the dashed arrow exists in any diagram as above with f a morphism in F.

Definition 32. A **model structure** on a category $\mathcal{C}$ is given by three classes of morphisms of $\mathcal{C}$ which are called **fibrations**, **cofibrations**, and **weak equivalences**, such that the following is satisfied:

(4.1) Any morphism f can be factored as $f = pi$ with i a cofibration and weak equivalence and p a fibration, as well as $f = pi$ with i a cofibration and p a fibration and weak equivalence.

(4.2) if for any tuple (f, g) of composable morphisms two out of f, g, and $g \circ f$ is a weak equivalence, then so is the third.

(4.3) Any two of the classes of fibrations, cofibrations and weak equivalences determines the third, in the way that a morphism:

- (a) is a fibration iff it has the RLP with respect to morphisms which are both cofibrations and weak equivalences
- (b) is a cofibration iff it has the LLP with respect to morphisms which are both fibrations and weak equivalences
- (c) is a weak equivalence iff it can be written as $u \circ v$ with v having the LLP with respect to fibrations and u having the RLP with respect to cofibrations.

Definition 33. A **model category** is a category with all finite limits and colimits together with a model structure on it.

Definition 34. A closed symmetric monoidal category endowed with a model category structure is a **monoidal model category** if it satisfies the following axiom: for any pair of cofibrations $f\colon A \to A'$ and $g\colon B \to B'$ the induced map $A \otimes B' \sqcup_{A\otimes B} A' \otimes B \to A' \otimes B'$ is a cofibration and it is additionally a weak equivalenve if so is one of f or g.

A well known fact that we will use is the following:

Lemma 35. Cofibrations are closed under pushouts, that is to say, given two morphisms $i\colon A \to B$ and $f\colon A \to C$, if $i\colon A \to B$ is a cofibration then so is $C \to B \sqcup_A C$.

Proof. This is an easy result which uses the universal property of the coproduct and the LLP of the cofibration.

□

5. A model structure on the category of topological operads

5.1. Compactly generated spaces. It is well known that the category of topological spaces together with continuous maps is too big for the purposes of homotopy theory, since it does not have many properties which a good homotopy theory should have [**Ste67**].

On the other hand, its full subcategory of compactly generated spaces is a convenient category from the point of view of homotopy theory [**Wyl73**], [**Str09**]. We next recall the definition of compactly generated spaces.

Definition 36. A topological space X is **weak Hausdorff** iff for every continuous map $f : K \to X$ where K is a compact Hausdorff space, the image $f(K)$ is closed in X. A subset U of a topological space X is **compactly open** iff for all continuous maps $f : K \to X$ where K is compact, the preimage $f^{-1}(U)$ is open in K. A space in which every compactly open subset is open is called a ***k*-space**. A **compactly generated space** is a weak Hausdorff k-space.

We denote the full subcategory of the category of topological spaces and continuous maps consisting of compactly generated spaces by **Top**. This category has the structure of a closed symmetric monoidal category and it is complete and cocomplete.

For the purpose of this paper we do not incur in a restriction by working only with compactly generated topological spaces, since the spaces $\overline{\mathcal{M}}_{0,n}$ which we define in Section 9 are compact spaces and as thus are compactly generated. From now on a topological space will mean a compactly generated topological space.

There is a model structure on Top in which weak equivalences are weak homotopy equivalences and fibrations are Serre fibrations. This model structure is called **Serre model structure** (or **Quillen model structure**).

The idea is now to transfer this model structure to the categories in the following diagram, where only the forgetful functors are drawn, using the respective adjunctions:

$$\begin{array}{ccccc} \mathrm{Op} & \longrightarrow & \Sigma\text{-}\mathrm{Top}_\star & \longrightarrow & \Sigma\text{-}\mathrm{Top} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{nsOp} & \longrightarrow & \mathbb{N}\text{-}\mathrm{Top}_\star & \longrightarrow & \mathbb{N}\text{-}\mathrm{Top} \end{array}$$

Crans was the first [**BM02**, Section 2.5] to describe this process in full generality: [**Cra95**, Section 3] suppose we are given categories $\mathcal{C}$ and $\mathcal{D}$ and adjoint functors $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ between them and that $\mathcal{C}$ has the structure of a model category. Then under certain conditions on $\mathcal{C}$, on $\mathcal{D}$ and on the adjunction, it is possible to define a model structure on the category $\mathcal{D}$ in which a morphism f

- is a weak equivalence in $\mathcal{D}$ iff $U(f)$ is a weak equivalence in $\mathcal{C}$
- is a fibration in $\mathcal{D}$ iff $U(f)$ is a fibration in $\mathcal{C}$, or equivalently iff it has the right lifting property with respect to all morphisms $F(g)$, where g is a trivial cofibration in $\mathcal{C}$
- is a cofibration in $\mathcal{D}$ iff it has the left lifting property with respect to all trivial fibrations in $\mathcal{D}$.

Using a similar transfer principle, one can transfer the model structure of Top to the model structure on G-spaces [**BM02**, Section 3] and the model structure on the category of symmetric collections to the category of operads [**BM02**, Theorem 3.2]. We will not go into the details of this transfer principle here; they can be found in [**BM02**], and in part also in [**Cra95**].

5.2. ***G*-spaces.**

Definition 37. A topological space together with the action of a discrete group G is called a ***G*-space**.

We denote the closed symmetric monoidal category of G-spaces together with G-equivariant continuous maps between them by ***G*-Top**. There is an adjunction $F\colon \mathrm{Top} \leftrightarrows G\text{-}\mathrm{Top}\colon U$ in which the left adjoint sends a space X to the G-space given by X together with free G-action. The model structure on Top can be transferred to G-Top in such a manner that a morphism f is a fibration (resp. a weak equivalence) in G-Top iff $U(f)$ is a fibration (resp. a weak equivalence) in Top [**BM02**, Section 3]. Recall that in any model category an object is called **cofibrant** (resp. **fibrant**) if the morphism from the initial object (resp. to the terminal object) is a cofibration (resp. a fibration). It is a well known fact that in the Serre model structure every object is fibrant, hence also every G-space is fibrant in the induced model structure.

Definition 38. We call a G-space for which the action of the group G is free a **free *G*-space**.

A necessary condition for a G-space to be cofibrant is that it is a free G-space:

Proposition 39. Let X be a cofibrant G-space. Then the action of G on X is free.

Proof. For every topological group G there exists a free G-space EG which is contractible [**Str11**, Theorem 16.48]. Consider the following diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & EG \\ {\scriptstyle f}\downarrow & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle g} \\ X & \longrightarrow & \star \end{array}$$

where f is a cofibration by assumption, g is a weak equivalence, since EG is contractible, and a fibration, since every object in G-Top is fibrant. Hence there exists a G-equivariant continuous map ψ which makes the diagram commute. Now, if for some $g \in G$ and $x \in X$ we have $gx = x$, then $\psi(gx) = g\psi(x) = \psi(x)$ and since the action of G on EG is free, g must be the neutral element of G. Thus X is a free G-space. $\square$

5.3. Collections. The model structure on Top induces a model structure on the category of collections, which is the product of monoidal categories $\prod_{n\geq 0}$ Top, in which a morphism is a cofibration, fibration or weak equivalence iff it is so entrywise in Top.

In an analogous fashion, if we consider the symmetric groups Σ_n, $n \geq 0$, the model structure on Σ_n-Top induces a model structure on the product of categories $\prod_{n\geq 0} \Sigma_n$-Top, in which a morphism is a cofibration, fibration or weak equivalence iff it is so entrywise in Σ_n-Top, for every $n \geq 0$. We denote this category by **Σ-Top** and call its objects **symmetric collections**.

5.4. Pointed spaces. We denote the category of pointed topological spaces together with continuous maps preserving the basepoints by $\mathbf{Top}_\star$. If $\mathcal{C}$ is any model category with terminal object $\star$, and $\mathcal{C}_\star$ denotes the coslice category $\star \downarrow \mathcal{C}$, then there is an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{C}_\star : U$ in which F sends an object X of $\mathcal{C}$ to the coproduct $X \sqcup \star$. The category $\mathcal{C}_\star$ inherits a model structure in which a morphism f is a cofibration, (resp. fibration, resp. weak equivalence) iff $U(f)$ is a cofibration, (resp. fibration, resp. weak equivalence) in $\mathcal{C}$ [**Hov07**, Proposition 1.1.8]. In particular, for $\mathcal{C} =$ Top there is a model structure on $\text{Top}_\star$ in which a morphism is a cofibration, (resp. fibration, resp. weak equivalence) iff it is so in Top.

Definition 40. An object of a category with terminal object is **pointed** if there exists a morphism from the terminal object to it. Let $X \in \text{Top}_\star$ (resp. Top). Define X to be **well-pointed** iff the morphism $\{\star\} \to X$ is a cofibration in Top (resp. iff X is pointed and the morphism $\{\star\} \to X$ is a cofibration).

We have:

Proposition 41. X in $\text{Top}_\star$ is cofibrant iff it is cofibrant in Top and well-pointed.

Proof. Remark that X is cofibrant in $\text{Top}_\star$ iff the morphism from the initial object $\{\star\}$ of $\text{Top}_\star$ to X is a cofibration, iff X is well-pointed. This implies that X is cofibrant in Top, since the terminal object of Top is cofibrant. $\square$

5.5. Pointed collections. A symmetric collection C in Σ-Top is **pointed** iff there is a morphism $\{\star\} \to C(1)$, it is **well-pointed** iff the morphism $\{\star\} \to C(1)$ is a cofibration in Σ-Top. We write **Σ-Top$_\star$** for the category of pointed symmetric collections together with equivariant morphisms respecting the basepoint. This category is the coslice category $\{\star\} \downarrow \Sigma$-Top and as thus inherits a model structure from Σ-Top in which a morphism is a cofibration (resp. fibration, resp. weak equivalence) iff it is so in Σ-Top. The initial object in Σ-Top$_\star$ is given by the symmetric pointed collection consisting of $\{\star\}$ in arity 1 and the empty set in all other arities. As a corollary of Proposition 41 we obtain:

Corollary 42. A pointed collection C is cofibrant in Σ-Top$_\star$ iff it is cofibrant in Σ-Top and well-pointed.

5.6. Operads. Berger and Moerdijk showed [**BM02**, Theorem 3.2] that the model structure on symmetric collections can be transferred to the category of (symmetric) operads in Top in such a way that a map $\mathcal{P} \to \mathcal{Q}$ is a weak equivalence (resp. a fibration) of operads iff for each n the map $\mathcal{P}(n) \to \mathcal{Q}(n)$ is a weak equivalence (resp. a fibration) in Top.

Definition 43. An operad is **Σ-cofibrant** if the underlying symmetric collection is cofibrant and it is **well-pointed** if the unit morphism is a cofibration.

Every cofibrant operad is Σ-cofibrant [**BM02**, Proposition 4.3] and hence by Proposition 39 for each $n \geq 1$ the action of the symmetric group Σ_n on $\mathcal{P}(n)$ is free.

6. The W-construction

The W-construction was introduced by Boardman and Vogt [**BV73**] to study homotopy invariant algebraic structures on topological spaces.

Definition 44. For an object C in a model category a **cofibrant replacement** is a cofibrant object C' together with a weak equivalence $C' \to C$.

Berger and Moerdijk showed that if $\mathcal{P}$ is a Σ-cofibrant and well-pointed operad, W($\mathcal{P}$) gives a cofibrant replacement of $\mathcal{P}$ in the model category introduced in the previous section [**BM05**, Theorem 5.1] and further they showed that in this case structures of algebras over W($\mathcal{P}$) are invariant under homotopy in the sense of Boardman and Vogt [**BM02**, Theorem 3.5].

In the following we recall the W-construction as defined by Boardmann and Vogt and give explicit calculations of W(As) and W(Comm).

Given an operad $\mathcal{P}$, define a new operad W($\mathcal{P}$), where for $n \geq 0$ an element of W($\mathcal{P}$)(n) consists of the following data:

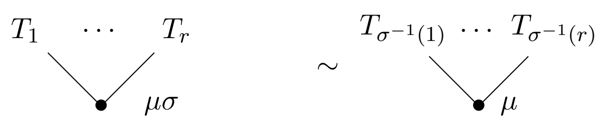
(6.1) a $U(\mathcal{P})$-tree T

(6.2) a map $h : E(T) \to [0, 1]$ such that the leaves and the root have length 1.

This data is subject to the following relations:

(6.1) for any tree (T, h) in $\mathrm{W}(\mathcal{P})(n)$, any vertex labelled by $id_{\mathcal{P}}$ together with the two adjacent edges labelled by t_1 and t_2, is equivalent to an edge labelled by $t_1 \star t_2 = t_1 + t_2 - t_1 t_2$.

(6.2) for any tree (T, h) in $\mathrm{W}(\mathcal{P})(n)$, any subtree of T formed by a vertex v in T of arity r labelled by $\mu\sigma$, $\sigma \in \Sigma_r$ and $\mu \in \mathcal{P}(r)$, together with all directed paths ending in v, which we can consider as subtrees $T_1, \ldots, T_r$, is equivalent to the subtree of T in which v is labelled by μ and the directed paths ending in v are the subtrees $T_{\sigma^{-1}(1)}, \ldots, T_{\sigma^{-1}(r)}$.



(6.3) any edge of length 0 may be shrunk away by composing its vertices using the composition in $\mathcal{P}$.

Suppose that λ is the underlying graph of a tree. Then the subspace of $\mathrm{W}(n)$ determined by λ is determined by the labels of the vertices and those of the internal edges as well as the symmetries, so we have

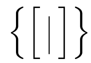
$$\mathrm{W}(\mathcal{P})(n) = \Big(\coprod_\lambda \Big(\prod_{j=0}^{r_\lambda+n} \mathcal{P}(j)^{m_\lambda(j)} \times [0,1]^{r_\lambda}\Big) \times \Sigma_n\Big)/\sim$$

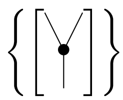
where r_λ is the number of internal edges of λ and $m_\lambda(j)$ is its number of vertices of arity j; further Σ_n is endowed with the discrete topology, the quotient is given by the above equivalence relations and we assign to $\mathrm{W}(\mathcal{P})(n)$ the quotient topology.

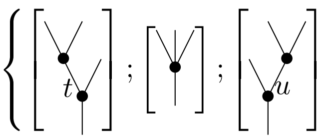
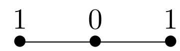
Composition is defined by grafting trees and assigning length 1 to the new internal edge. The unit for this composition is given by the 1-tree without vertices and unique edge labelled by 1. There is an obvious right action of Σ_n on $\mathrm{W}(\mathcal{P})(n)$ defined by $[(T, h)] \mapsto [(T\sigma, h)]$ for all $\sigma \in \Sigma_n$ and $[(T, h)] \in \mathrm{W}(\mathcal{P})(n)$. Composition and the action of the symmetric groups are easily seen to be well-defined, so with this data we have that $\mathrm{W}(\mathcal{P})$ is a symmetric topological operad.

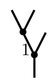
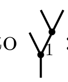
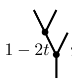
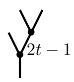
Remark 45. For non-symmetric operads, the W-construction differs from the symmetric case in that we consider planar trees such that each tree with n leaves has the leaf labels $1, 2, \ldots, n$, from left to right. Therefore leaf labels can be omitted. The relations are (6.1) and (6.3).

Example 46. Consider the non-symmetric operad As_+ whose algebras are topological semigroups, namely for each $n \geq 1$ we have $\mathrm{As}_+(n) = \{\star\}$, the one-point set. Then:

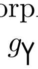
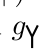
$$\mathrm{W}(\mathrm{As}_+)(1) = \Big\{$$  $$\Big\} \cong \{\star\}$$

$$\mathrm{W}(\mathrm{As}_+)(2) = \Big\{$$  $$\Big\} \cong \{\star\}$$

$$\mathrm{W}(\mathrm{As}_+)(3) = \Big\{$$  $$\Big| \; t, u \in [0,1] \Big\} \cong$$ 

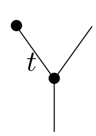
Thus $\mathrm{W}(\mathrm{As}_+)(3)$ is a subdivided interval; it gives a path from  to  to : with $\gamma(t) =$ , $t \in [0, \frac{1}{2}]$ and $\gamma'(t) =$ , $t \in [\frac{1}{2}, 1]$ the path is given by $\gamma' \circ \gamma$.

We drop the square brackets to facilitate notation. Analogously, $\mathrm{W}(\mathrm{As}_+)(4)$ is a cell decomposition of the pentagon K_4. In general, $\mathrm{W}(\mathrm{As}_+)(n)$ is a cell decomposition of the n-th Stasheff polytope K_n [**LV12**, Appendix C2.2]. Loday and Vallette showed that a special kind of planar trees, called circled planar trees, encode these cell decompositions of the Stasheff polytopes [**LV12**, Appendix C2.3].

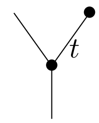
Let A be an algebra over $\mathrm{W}(\mathrm{As}_+)$. Then, since Top is a closed symmetric monoidal category it makes sense to speak about endomorphism operads (see Appendix A) and thus we have a morphism $\phi : \mathrm{W}(\mathrm{As}_+) \to \mathrm{End}_A$. We write g_T for $\phi(n)(T)$. Then $g_|$ is the identity on A, whilst g is a morphism g $: A \times A \to A$.

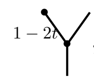
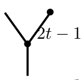
The path $\gamma' \circ \gamma$ induces a homotopy $f_t : g$ $\to g$, where $f_t = \phi(3)(\gamma' \circ \gamma(t))$, hence we obtain that g $\circ_1 g$ is homotopic to g $\circ_2 g$ and therefore A is a topological semigroup in which associativity holds only up to homotopy. Similarly, the pentagon K_4 gives two homotopies between g and g and the cell subdivision gives a homotopy between these two homotopies, and so on for every n up to infinity.

Example 47. Now we consider the operad As encoding topological monoids which is given by $\mathrm{As}(n) = \{\star\}$ for all $n \geq 0$. The space $\mathrm{W}(\mathrm{As})(0)$ is a countably infinite space: its operations are given by trees with only dead leaves. Similarly, also all spaces $\mathrm{W}(\mathrm{As})(n)$ for $n \geq 1$ are infinite. In particular, in $\mathrm{W}(\mathrm{As})(1)$ there are the operations



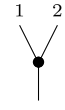
and

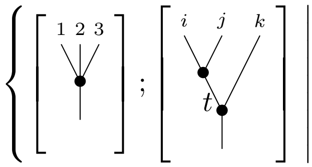


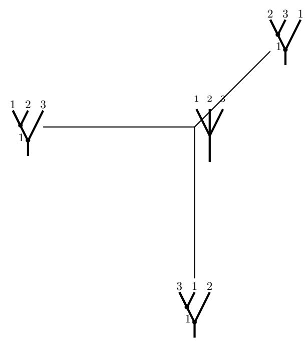
There is a path $\gamma' \circ \gamma\colon [0,1] \to \mathrm{W(As)}(1)$ with $\gamma(t) =$ , $t \in [0, \frac{1}{2}]$ and $\gamma(t) =$ , $t \in [\frac{1}{2}, 1]$ and $\gamma' \circ \gamma(\frac{1}{2}) =$ $|$. By a similar argument as used in the previous example, this path induces a homotopy from $g_{Y} \circ_1 g$ to $g_{Y} \circ_2 g$ and therefore A is a topological monoid in which associativity and unitality hold only up to homotopy. Remark that associativity holds up to higher coherence conditions, while no such statement is made about unitality. The analogous of the subdivision of the Stasheff polytope for unital associahedras was constructed in recent work by Muro and Tonks [**MT11**].

Example 48. The symmetric operad Comm, whose algebras are commutative topological semigroups, is given in each arity $n \geq 1$ by $\mathrm{Comm}(n) = \{\star\}$, the one point set, together with the trivial action of the symmetric group. To calculate the W-construction for Comm we need to use planar trees with labelled leaves. There are $(2n-3)!!$ labelled binary trees with n leaves, hence up to non-planar isomorphisms, there are as much labelled binary planar trees with n leaves (and no labels on the vertices). We have:

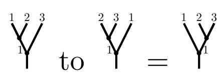
$$\mathrm{W(Comm)}(1) = \left\{ \left[\, | \, \right] \right\} \cong \{\star\}$$

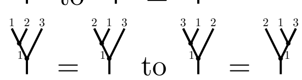
$\mathrm{W(Comm)}(2) = \Big\{ \Big[$  $\Big] \Big\} \cong \{\star\}$

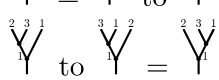
W(Comm)(3) =  $t \in [0,1]$ and $(i,j,k) \in \{(1,2,3),(2,3,1),(3,1,2)\}$

$\cong$ 

There are 15 labelled binary planar trees with 4 leaves. The space $W(\mathrm{Comm})(4)$ is given by the polytope obtained from gluing together 15 quadrants along their common edges. Let A be an algebra over W(Comm) and consider the morphism $\phi : \mathrm{W(Comm)} \to \mathrm{End}_A$. Similarly to the previous example, we write g_T for $\phi(n)(T)$. Again, we have that $g_|$ is the identity on A and g_{Y} is a morphism $g_{\mathsf{Y}} : A \times A \to A$. Now, we can choose the following three different paths in W(Comm)(3) passing through the corolla  exactly once:

(6.1) from 

(6.2) from  and

(6.3) from 

By the same arguments of Example 46, we thus see that all three paths give associativity up to homotopy. Analogously, the 15 binary trees of W(Comm)(4) give several paths from the left to the right comb with same labelling and homotopies between them, and so on up to infinity. Thus A is associative up to strong homotopy. It is however only strictly commutative, the action of the symmetric group on Comm being trivial.

Remark 49. There is a model structure on Top in which weak equivalences are homotopy equivalences and fibrations are Hurewicz fibrations. This model structure is called **Hurewicz model structure**, or also **Strøm model structure**, since Strøm was the first to prove its existence in [**Sm72**]. The cofibrancy of $W(\mathcal{P})$ for a Σ-cofibrant well-pointed operad $\mathcal{P}$ with respect to the Hurewicz model structure was proved by Vogt [**Vog03**].

Tropical moduli spaces

Tropical geometry is an area of mathematics which has received a lot of attention in the past 20 years or so due to its relation to algebraic geometry.

In the words of Sturmfels and Speyer [**SS04**] the adjective tropical *'simply stands for the French view of Brazil.'*; indeed, French mathematicians baptized min-plus semirings - semirings in which the operations are given by taking addition and minimum on certain sets - "tropical semirings", in honour of the Brazilian mathematician Imre Simon who was a pioneer in the field [**Pin98**], [**Sim88**].

Nowadays by tropical semiring one usually understands the min-plus or max-plus semiring, whose underlying set is the set of real numbers and with operations given by

$$\oplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto \max\{a, b\}$$

and

$$\odot : \mathbb{R} \times \mathbb{R} \to \mathbb{R} : (a, b) \mapsto a + b,$$

where $+$ denotes the usual addition of real numbers. Then $(\mathbb{R}, \oplus)$ is a commutative semigroup and $(\mathbb{R}, \odot)$ is a commutative group with neutral element 0. If we extend the set of real numbers by an element $-\infty$ and require

$$-\infty \oplus r = r \oplus -\infty = r \quad (r \in \mathbb{R})$$

$$-\infty \oplus -\infty = -\infty,$$

we obtain the commutative monoid $(\mathbb{R} \cup \{-\infty\}, \oplus)$ with neutral element $-\infty$. If we further impose

$$-\infty \odot r = r \odot -\infty = -\infty \quad (r \in \mathbb{R} \cup \{-\infty\}),$$

then $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$ is an idempotent semifield called the **tropical semifield**, often denoted by $\mathbb{R}_{tr}$. These operations can be extended componentwise to $\mathbb{R} \cup \{-\infty\} \times \cdots \times \mathbb{R} \cup \{-\infty\}$ and thus we also obtain an idempotent semifield $\mathbb{R}^n_{tr}$. We will also use $\mathbb{R}^n_{tr}$ to denote the underlying set of this semifield. Tropical mathematics usually means mathematics involving the tropical semifield.

Analogously to the classical case, one can define a semiring $\mathbb{R}_{tr}[X_1, \ldots, X_n]$ whose elements are polynomials with coefficients in $\mathbb{R}_{tr}$ and are called **tropical polynomials**. This semiring satisfies a universal property analogous to the universal property of the polynomial

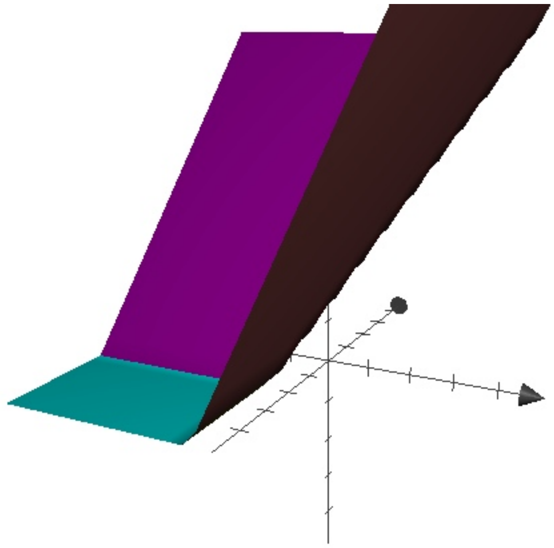


FIGURE 1. Convex piecewise linear graph of $1 \odot X^2 \oplus 2 \odot Y \oplus 0$.

ring with coefficients in a ring. The evaluation of a tropical polynomial gives a convex piecewise linear subset of $\mathbb{R}^n_{tr}$. For example, for the polynomial $f(X,Y) = 1 \odot X^2 \oplus 2 \odot Y \oplus 0$ the graph of the function $ev(f)\colon \mathbb{R}^2_{tr} \to \mathbb{R}_{tr}\colon (x,y) \mapsto f(x,y)$ is the convex subset of $\mathbb{R}^3_{tr}$ of Figure 1.

A first attempt at defining a tropical curve could be to try to evaluate at $-\infty$, but this does not give anything interesting, since if a_0 denotes the constant term of the polynomial we have $ev(f)(x) \geq a_0$ for all $x \in \mathbb{R}^2_{tr}$, so the equality $ev(f)(x) = -\infty$ has solutions only if $a_0 = -\infty$. The unique solution is then $x = (-\infty, -\infty)$.

However, there is another equivalent characterization of roots of a polynomial in one variable in classical algebra and this characterization turns out to be a good approach for tropical algebra, since it gives the following:

FUNDAMENTAL THEOREM OF TROPICAL ALGEBRA. [**GM07**][**BS13**, Proposition 1.1] Every tropical polynomial in $\mathbb{R}_{tr}[X]$ of degree d has exactly d roots counted with multiplicities.

But we haven't defined what the degree and roots are yet. The degree is defined analogously as for classical algebra. For the root we have:

Definition 50. A point $x \in \mathbb{R}_{tr}$ is a **root** of the tropical polynomial $f \in \mathbb{R}_{tr}[X]$ iff there exists a tropical polynomial g such that $f = (x \oplus x_0) \odot g$.

It is easy to see that the roots of a tropical polynomial in one variable are the corners of the convex piecewise linear graph. For a tropical polynomial in one variable these are exactly the points at which at least two monomials attain the maximum. If we write $f = \bigoplus_{i \in I \subset \mathbb{Z}} a_i \odot X^i$, then the corners are given by those points x for which there exist $i \neq j \in I$ such that

$$ev(f)(x) = a_i \odot x^i = a_j \odot x^j. \tag{0.1}$$

Definition 51. The **multiplicity** of a root is defined to be the difference in the slopes of the two pieces adjacent to a corner.

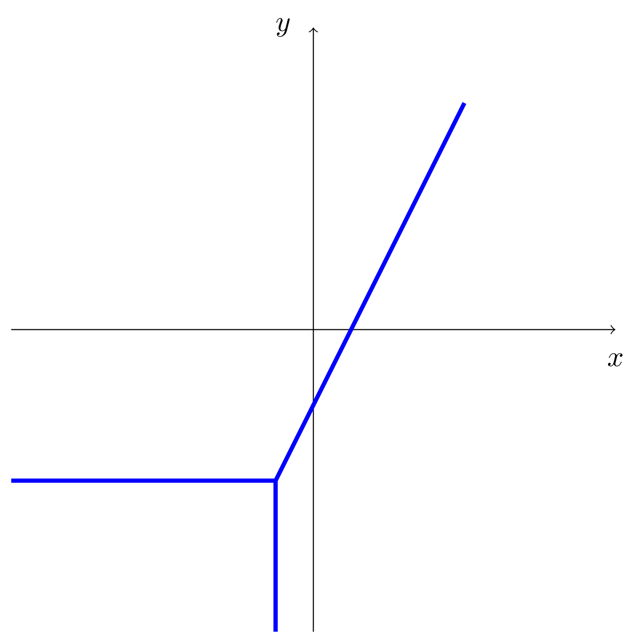
This is the maximum of $|i - j|$ for all $i \neq j \in I$ for which (0.1) is satisfied.

Now, if one wants to carry this over to a polynomial in two variables, one could define a tropical curve as the corner locus of the graph of the tropical polynomial, i.e. those points at which the maximum is attained at least twice.

For example, the corners of the polynomial $f = 1 \odot X^2 \oplus 2 \odot Y \oplus 0$ are the solutions to the three systems of inequalities

$$1 + 2x = 2 + y \geq 0 \iff y = 2x - 1 \text{ and } x \geq -\frac{1}{2}$$
$$1 + 2x = 0 \geq 2 + y \iff x = -\frac{1}{2} \text{ and } y \leq -2$$
$$2 + y = 0 \geq 1 + 2x \iff y = -2 \text{ and } x \leq -\frac{1}{2}$$

which give the following subset of $\mathbb{R}^2_{tr}$:

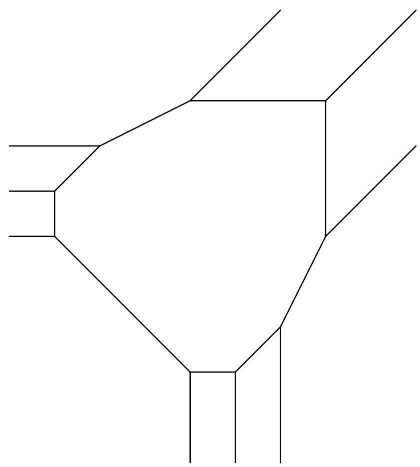


One is thus lead to give the following definition which turns out to be a good choice (see later).

Definition 52. The **(affine[2]) tropical curve** defined by a polynomial $f = \bigoplus_{i \in I \subset \mathbb{Z}^2_{\geq 0}} a_i \odot X^{i_1} \odot Y^{i_2}$ in $\mathbb{R}_{tr}[X, Y]$ is the set of points (x, y) of $\mathbb{R}^2_{tr}$ for which there exist $i \neq j \in I$ such that

$$ev(f)(x, y) = a_i \odot x^{i_1} \odot y^{i_2} = a_j \odot x^{j_1} \odot y^{j_2}.$$

The genus of such a curve is what one would expect from classical geometry, it is the number of cycles. For example the following is a tropical curve of genus 1:



The degree however is not given by the degree of the polynomial, to define it we need the concept of Newton polygon associated to the curve (see later).

Analogously to the univariate case, for every corner edge e we have to take into account differences of the slopes on the two sides ot the edge. One thus assigns a weight to every edge:

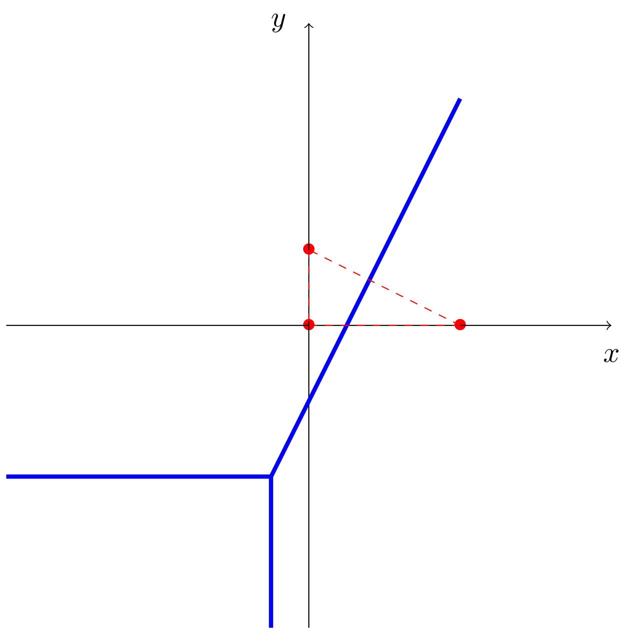
Definition 53. The **weight** of an edge e of a tropical curve is:

$$w(e) = \max_{i,j \text{contributing to } e} (gcd(|i_1 - j_1|, |j_1 - j_2|)).$$

The vertex of the curve determined by the polynomial $f = 1 \odot X^2 \oplus 2 \odot Y \oplus 0$ is given by the point where all three monomials are equal, which is $(-\frac{1}{2}, -2)$. If we take the convex hull of the exponents $(2, 0), (0, 1), (0, 0)$ of f, then it turns out that the edge between two

[2]There is a projective tropical space $\mathbb{P}^2_{tr}$ and projective tropical curves are defined as corner loci of homogeneous tropical polynomials. There is a similar characterization as the one for affine tropical curves of Proposition 54.

exponents is perpendicular to the edge of the curve that they determine, as illustrated in the following picture:



If for every edge e_i adjacent to $(-\frac{1}{2}, -2)$ we take the primitive integer vector v_i at $(-\frac{1}{2}, -2)$ along this edge, then as a consequence of the fact that the polygon is closed we have

$$\sum_i w(e_i)v_i = 0.$$

More generally, to every tropical curve we can assign a polytope which is given by taking the convex hull of its support and is called **Newton polytope**. This polytope has a subdivision which is dual to the tropical curve:

- boundary edges correspond to unbounded edges of the tropical curve
- internal edges correspond to bounded egdes of the curve
- 2-cells correspond to vertices of the curve.

As a consequence of this duality, a tropical curve satisfies at every vertex the following **balancing condition**: let v be any vertex of the curve and suppose that it has adjacent edges $e_1, \ldots, e_m$. Let v_i be a primitive integer vector starting at v and pointing in direction e_i. Then

$$\sum_{i=1}^{m} w(e_i)v_i = 0.$$

So every tropical curve is a weighted graph, with edges having rational slopes, and which satisfies the balancing condition at every vertex. The converse is also true:

Proposition 54. The tropical curves in $\mathbb{R}^2$ are the rational weighted graphs satisfying the balancing condition at every vertex.

Proof. See [**Mik03**, Corollary 3.16]. □

There is no consensus in the literature on how to define the degree of a tropical curve. A possibility is to define the degree of the curve in terms of its Newton polytope. However, also here there are at least two different definitions of degree: some authors say that a curve has **degree *d*** if its Newton polytope is up to a translation the d-dimensional symplex in $\mathbb{R}^2$. Others give a weaker definition: a curve has **degree *d*** if its Newton polytope is up to translation a subset of the d-dimensional symplex, and the curve is said to have **degree *d* with full support** if the Newton polytope is up to translation the d-dimensional symplex. Here we will use the first definition, since we use results by authors who rely on it.

Let us finish this introduction with a quick explanation of the original motivation for studying tropical geometry. For every algebraic curve we may obtain a tropical curve by a so-called dequantization process, and the tropical curve is easier to understand than the original curve, due to its combinatorial nature.

Consider the family of semifields $\{\mathbb{R}_t\}_{t>0}$ with as underlying set the real numbers and the following operations:

$$a \oplus_t b = t(log(e^{\frac{a}{t}} + e^{\frac{b}{t}}))$$

$$a \odot_t b = a + b$$

Each semifield $\mathbb{R}_t$ is isomorphic to the semifield $\mathbb{R}_+$ given by the set of non-negative real numbers together with the usual addition and multiplication. If we take the limit $\lim_{t\to 0} a \oplus_t b = a \oplus b$ then we obtain a semifield $\mathbb{R}_0$ which is not isomorphic to $\mathbb{R}_+$. This is our tropical semifield. The passage to the limit $\lim_{t\to 0} a \oplus_t b$ is called **Maslov dequantization of the real numbers** [**Lit05**].

Just as the tropical semiring is the limit of $\mathbb{R}_+$, tropical curves are limits of classical curves. Define the map

$$Log\colon (\mathbb{C}^\star)^2 \to \mathbb{R}^2\colon (x,y) \mapsto (\log(|x|), \log(|y|))$$

for a complex curve C, the image of its restriction to $(\mathbb{C}^\star)^2$ under Log is called **amoeba** of C. Now consider the following map

$$Log_t\colon (\mathbb{C}^\star)^2 \to \mathbb{R}^2\colon (x,y) \mapsto (-\frac{\log(|x|)}{\log t}, -\frac{\log(|y|)}{\log t})$$

Taking the limit as t goes to 0 we shrink the width of the tentacles of the amoeba to zero and obtain a combinatorial object. If instead of a single curve one considers a family of curves $\{C_t\}_{t>0}$, then the limit of the amoeba $Log_t(C_t \cap (\mathbb{C}^\star)^2)$ as t tends to zero is a tropical curve as defined in Definition 52 [**BS13**].

7. Abstract tropical curves as metric graphs

From what we have seen in the previous section, it makes sense to think of an abstract tropical curve as a weighted balanced rational graph G which is embedded in $\mathbb{R}^2_{tr}$ or $\mathbb{P}^2_{tr}$. More precisely, one can define an abstract tropical variety as a topological space together with a tropical structure given by charts, see [**Mik06**, Definition 3.1]. Compact tropical curves have a very simple characterization in terms of metric graphs, see [**MZ06**, Proposition 3.6] or [**Mik06**, Proposition 5.2]. Here we adopt this characterization as definition of abstract tropical curve.

Definition 55. A **graph** is a tuple $G = (V, E, s, t)$ where V is a non-empty finite set of **vertices**, E is a finite set of **edges**, and $s, t : E \to V$ are maps of sets. For any $e \in E$ we call $s(e)$ and $t(e)$ its **endpoints**. We also use the notation $s(e) \xrightarrow{e} t(e)$ for any $e \in E$. For any vertex v the cardinality of the set $\{e \in E \mid v \in t^{-1}(e)\} \sqcup \{e \in E \mid v \in s^{-1}(e)\}$ is called **valence** of v. An **isomorphism of graphs** $(V, E, s, t) \to (V', E', s', t')$ is a tuple (f_E, f_V) where $f_E \colon E \to E'$ and $f_V \colon V \to V'$ are bijections such that $s' \circ f_E = f_V \circ s$ and $t' \circ f_E = f_V \circ t$.

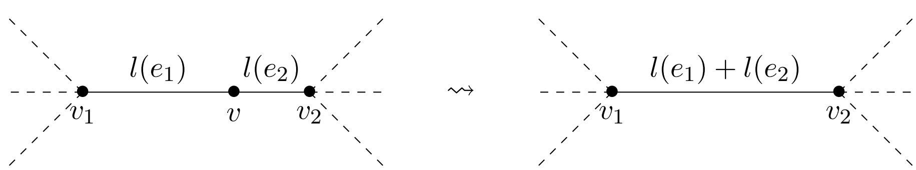
Definition 56. An edge adjacent to a 1-valent vertex of a graph is called **leaf**.

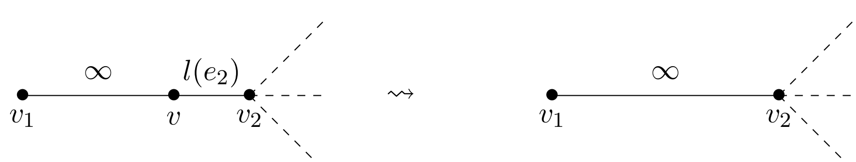
Definition 57. An **abstract tropical curve** is a connected graph G together with a map $l : E \to \mathbb{R}_{>0} \cup \{\infty\}$ so that $l(e) = \infty$ iff e is a leaf. An **isomorphism of abstract tropical curves** $(G, l) \to (G', l')$ is an isomorphism $\phi \colon G \to G'$ such that $l' \circ \phi = l$. The **genus** of an abstract tropical curve is the genus of its underlying graph.

We define the following equivalence relation on the set of abstract tropical curves.

Definition 58. Let (G, l) be an abstract tropical curve and let v be a 2-valent vertex of G adjacent to two edges e_1, e_2 with endpoints (other than v) v_1 and v_2 and lengths $l(e_1)$ and $l(e_2)$. The tropical curve obtained from (G, l) by removing v and substituting e_1 and e_2 with an edge e with endpoints v_1 and v_2 and length $l(e_1) + l(e_2)$, where we set $\infty + l = l + \infty = \infty$ for all $l \in (0, \infty]$, is a **reduction** of (G, l).

Here are two illustrations of such a move:





Definition 59. Two abstract tropical curves (C, l) and (C', l') are **isometrically equivalent** iff there exists a natural number $m \geq 0$ and abstract tropical curves $(C_0, l_0), \dots (C_m, l_m)$ such that $(C_0, l_0) = (C, l)$ and $(C_m, l_m) = (C', l')$ and for all $i = 1, \dots, m$ either (C_i, l_i) is a reduction of (C_{i-1}, l_{i-1}), or vice versa.

8. Tropical modifications and pointed curves

There is a feature of tropical geometry that makes it very different from classical geometry: in tropical geometry the shape of a curve depends on the space in which it lives; for this reason different tropical curves may be seen as models for the same classical variety. Luckily, there is a way to transform one model into another: there is an operation known as **tropical modification** which gives an equivalence relation on the set of tropical curves and relates different models to one another. For more on tropical modifications see [**Mik06**, Section 3.5] or [**BS13**, Section 6.2]. Here we give the definition of tropical equivalence for abstract tropical curves:

Definition 60. Let (G, l) be an abstract tropical curve. A **modification** of (G, l) is the abstract tropical curve obtained from (G, l) by removing a 1-valent vertex and the edge e adjacent to it.

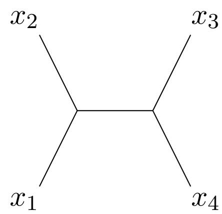
Definition 61. Two abstract tropical curves (C, l) and (C', l') of genus 0 are **tropically equivalent** iff there is a natural number $m \geq 0$ and abstract tropical curves $(C_0, l_0), \dots, (C_m, l_m)$ such that $(C_0, l_0) = (C, l)$ and $(C_m, l_m) = (C', l')$ and for all $i = 1, \dots, m$ either the curve (C_i, l_i) is a modification of (C_{i-1}, l_{i-1}), or vice versa.

Now, following an idea by Mikhalkin, if we have an abstract tropical curve together with $n \geq 2$ distinct points on it, which we call **marked points**, then there is a unique representant up to isomorphism in the tropical equivalence class of the curve in which the marked points coincide with the 1-valent vertices. Thus we can assume that the set of marked points coincides with the set of 1-valent vertices, and therefore the tropical analogue of an algebraic curve with n marked points is a tropical curve with n leaves. This motivates the following definition:

Definition 62. For $n \geq 2$ an ***n*-pointed tropical curve** of genus 0 is a triple (C, L, f) where C is an abstract tropical curve of genus 0 and L is a collection of n distinct leaves of C together with an isomorphism $f \colon L \to \{1, \dots, n\}$. The elements of L are called **marked points of *C***. The tuple (L, f) is a **marking** of C. An **isomorphism of *n*-pointed tropical curves** $(C, L, f) \to (C', L', f')$ is an isomorphism of abstract tropical curves $\alpha : C \to C'$ such that $\alpha(f^{-1}(i)) = f'^{-1}(i)$ for all $i = 1, \dots, n$. A **pointed tropical curve** is an n-pointed tropical curve for some $n \geq 2$. Two pointed tropical curves (C, L, f) and (C', L', f') are **tropically equivalent** if one can be obtained from the other by removing or adding a finite number of 1-valent vertices and the edges adjacent to them, so that these edges are neither in L nor in L'. Isometric equivalence of pointed tropical curves is defined as for unpointed tropical curves.

Remark that a tropical curve with n marked points has no nontrivial automorphisms.

For example, the 4-pointed curve



has 4 automorphisms as a tropical curve but just 1 automorphism of pointed tropical curve. Such curves can therefore be thought of as the tropical pendant to stable curves [**Cap11**].

9. Tropical moduli spaces

Definition 63. The set of equivalence classes of n-pointed tropical curves of genus 0 by the isometric and tropical equivalences is denoted by $\boldsymbol{\mathcal{M}_{0,n}}$.

Definition 64. The **combinatorial type** of a tropical curve is its underlying graph.

Each $\mathcal{M}_{0,n}$ is a disjoint union of subsets given by the different combinatorial types, and further $\mathcal{M}_{0,n}$ is a polyhedral complex, and as thus in particular a topological space [**Mik07**], [**GM05**, Example 2.13]. In particular, as a topological space, $\mathcal{M}_{0,n}$ is the space of metric $n-1$-trees examined by Billera, Holmes and Vogtmann in [**BHV01**].

Proposition 65. [**Mik07**, Theorem 1] For all $n \geq 3$ the moduli space $\boldsymbol{\mathcal{M}_{0,n}}$ is a tropical variety of dimension $n-3$.

The moduli space $\mathcal{M}_{0,n}$ is not compact, but it can be compactified [**Mik07**, Page 9], [**ACP12**, Section 4.1] by allowing also edges which are not leaves to have length ∞.

Definition 66. An **extended $\boldsymbol{n+1}$-pointed tropical curve** of genus 0 is a tuple (G, L, f, l) where $G = (E, V, s, t)$ is a graph of genus 0, together with a set L of $n+1$ distinct 1-valent vertices of G called **marked points**, an isomorphism $f\colon L \to \{0, 1, \dots, n\}$ and a map $l : E \to \mathbb{R}_{>0} \cup \{\infty\}$ such that $l(e) = \infty$ if e is a leaf. An **isomorphism of extended $\boldsymbol{n}$-pointed tropical curves** $(G, L, f, l) \to (G', L', f', l')$ is an isomorphism $\phi\colon G \to G'$ such that $f' \circ \phi = f$ and $l' \circ \phi = l$.

Isometric and tropical equivalence of extended pointed tropical curves is defined similarly as for pointed tropical curves.

For simplicity from now on we will denote such a curve by $\boldsymbol{(C, x_0, x_1, \dots, x_n)}$ with $C = (G, l)$ and $x_0, \dots, x_n \in L$ leaves of G indexed according to $x_i := f^{-1}(i)$ for $f\colon L \to \{0, 1, \dots, n\}$.

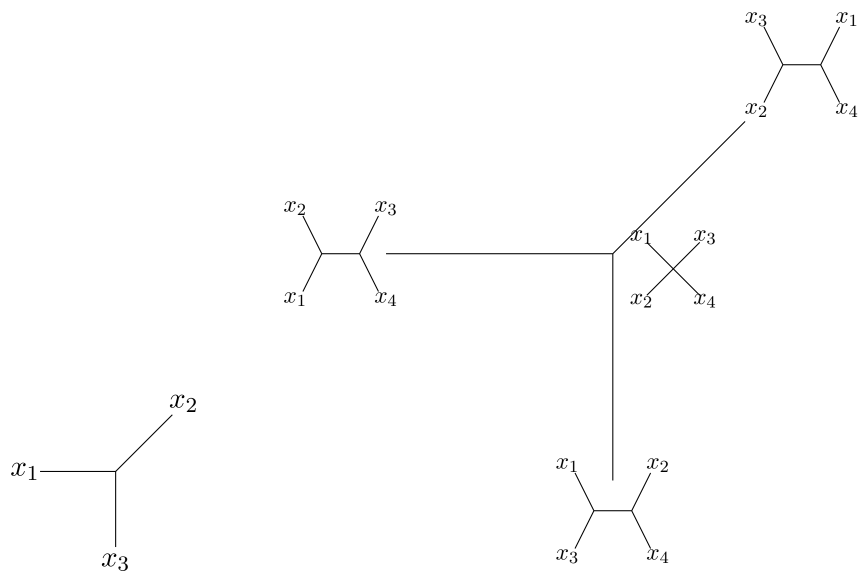


FIGURE 2. The moduli spaces $\mathcal{M}_{0,3}$ and $\mathcal{M}_{0,4}$

Definition 67. We denote by $\overline{\boldsymbol{\mathcal{M}}}_{\mathbf{0},\boldsymbol{n}}$ the set of equivalence classes of extended n-pointed tropical curves of genus 0 by the isometric and tropical equivalences.

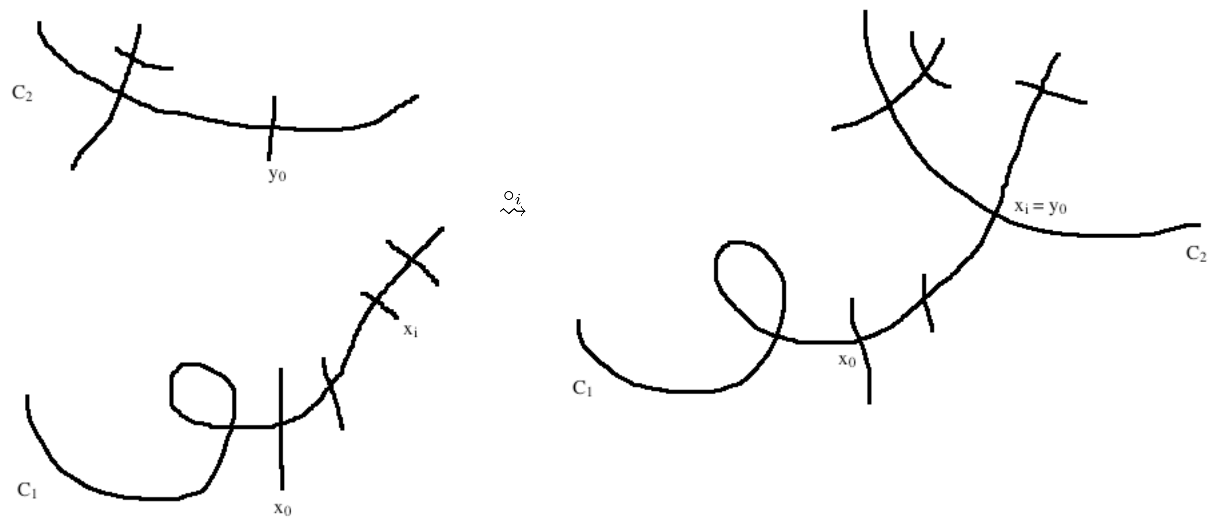
Mikhalkin showed [**Mik07**, Corollary 2.5] that for $n \geq 3$ $\overline{\mathcal{M}}_{0,n}$ is a smooth compact tropical variety. This compactification is the tropical analogue of the Deligne-Mumford compactification[3] [**Mik07**]. In the classical case $M_{0,n+1}$ is the moduli space of $(n+1)$-tuples $(x_0, \ldots, x_n)$ of distinct ordered points on $\mathbb{CP}^1$ modulo projective automorphisms. The compactification of $M_{0,n+1}$ is given by $\overline{M}_{0,n+1}$ which is the moduli space of stable $n+1$-pointed curves of genus 0.

The collection $\{\overline{M}_{0,n+1}\}_{n\geq 2}$ has the structure of a symmetric topological operad with composition

$$\circ_i \colon \overline{M}_{0,n+1} \times \overline{M}_{0,m+1} \to \overline{M}_{0,m+n} \quad (1 \leq i \leq n+1)$$

given by gluing two stable curves C_1 and C_2 at two nodal singularities [**GK07**, Section 1.4] as illustrated in the following picture:

[3]The Deligne-Mumford compactification is also called Grothendieck-Knudsen compactification in the literature. The reason is that Grothendieck and Knudsen were the first to construct the compactification for genus zero curves, while Deligne and Mumford generalized this to curves of higher genus [**MS12**, Page 651, Appendix D].



and then relabelling the marked points. See [**Mar06**] or [**GK07**] for more details.

Similarly to the classical case, the compactified moduli space can be given the structure of a topological operad. The composition is given by a modification of the **tropical clutching map** defined in [**ACP12**, Section 8.3]. In the tropical clutching map the last leaves of the curves are identified with one another. We can modify this definition slightly and define the following operadic composition:

Definition 68. For all $n \geq 2$ and all $1 \leq i \leq n+1$ define the map

$$\circ_i \colon \overline{\mathcal{M}}_{0,n+1} \times \overline{\mathcal{M}}_{0,m+1} \to \overline{\mathcal{M}}_{0,n+m}$$

$$\big([(C_1, x_0, \dots x_n)], [(C_2, y_0, \dots, y_m)]\big) \mapsto [(C, z_0, \dots, z_{m+n-1})]$$

where for $(C_1, x_0, \dots x_n) = (G_1, L_1, f_1, l_1)$ and $(C_2, y_0, \dots y_m) = (G_2, L_2, f_2, l_2)$ and edges $x_i \xrightarrow{e_i} v_1$ and $v_2 \xrightarrow{e_0} y_0$ the tropical curve $(C, z_0, \dots, z_{m+n-1})$ is given by:

- underlying graph $G = (V, E, s, t)$ where:
  - $V = V_1 \setminus x_i \sqcup V_2 \setminus y_0$
  - $E = E_1 \setminus e_i \sqcup E_2 \setminus e_0 \sqcup \{e'\}$
  - source and target of an edge are given by

$$\begin{array}{rccl} s\colon & E & \to & V \\ & e & \mapsto & \begin{cases} s_i(e), \text{ if } e \in E_i, i = 1, 2 \\ v_2, \text{ otherwise} \end{cases} \end{array}$$

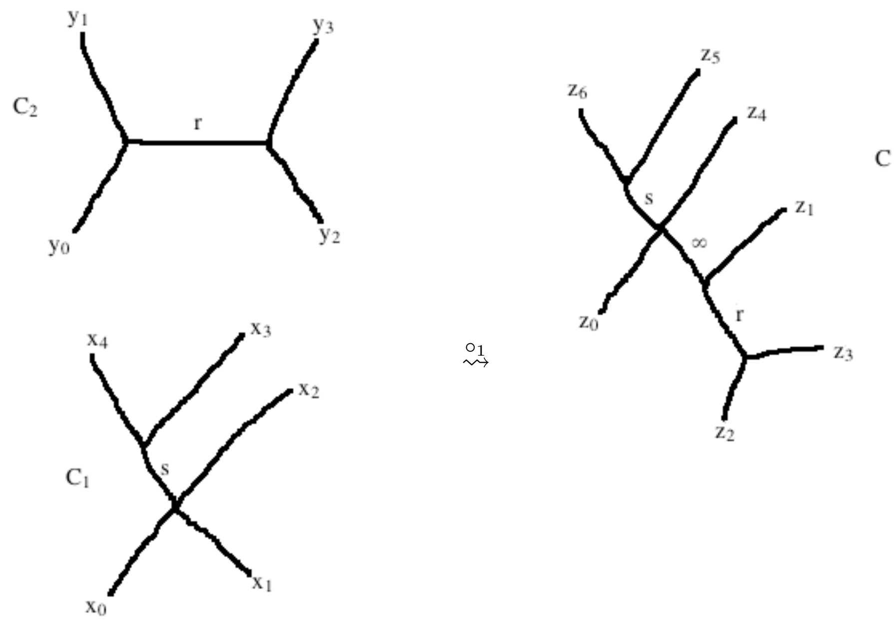


FIGURE 3. The tropical clutching map.

and

$$\begin{array}{rccl} t\colon & E & \to & V \\ & e & \mapsto & \begin{cases} t_i(e), \text{ if } e \in E_i, i = 1,2 \\ v_1, \text{ otherwise} \end{cases} \end{array}$$

- set of marked points the set of 1-valent vertices $L = L_1 \setminus x_i \sqcup L_2 \setminus y_0$
- bijection

$$\begin{array}{rccl} f\colon & L & \to & \{0,\dots,m+n-1\} \\ & v & \mapsto & \begin{cases} f_1(v), \text{ if } v \in L_1 \text{ and } 1 \le f_1(v) \le i-1 \\ f_2(v) + i - 1, \text{ if } v \in L_2 \\ f_1(v) + m, \text{ if } v \in L_1 \text{ and } f_1(v) \ge i \end{cases} \end{array}$$

- map $l\colon E \to \mathbb{R}_{>0} \cup \{\infty\}\colon e \to \begin{cases} l_i(e), \text{ if } e \in E_i \text{ for } i = 1,2 \text{ and } e \neq e' \\ \infty, \text{ otherwise.} \end{cases}$

We call this a **tropical clutching map**.

We next want to relate tropical moduli spaces of genus 0 to the W-construction of Comm. Boardman and Vogt used the monoid $([0,1],\star)$, while edges of the tropical moduli space trees are labelled by numbers in $(0,\infty]$.

Definition 69. Let $\boldsymbol{([0,\infty],+)}$ be the monoid with underlying set the one-point compactification of the non-negative real numbers, composition given by addition, where we set $\infty + r = \infty = r + \infty$ for all $r \in \mathbb{R}_{\geq 0}$ or $r = \infty$.

Since ∞ is an idempotent element of $([0,\infty],+)$ we can consider the edges of trees of the tropical moduli space as being labelled by elements of this monoid.

Lemma 70. [4] There is an isomorphism of topological monoids

$$\alpha : ([0,\infty],+) \to ([0,1],\star).$$

Proof. Consider the morphism of topological monoids

$$\phi : ([0,1],\star) \to ([0,1],\cdot) : x \mapsto 1-x.$$

We have

$$x \star y = 1 - (1-x)(1-y) = \phi^{-1}(\phi(x)\phi(y)),$$

and further the monoid $([0,1],\cdot)$ is isomorphic to $([0,\infty],+)$ via the morphism of topological monoids

$$([0,1],\cdot) \to ([0,\infty],+) : x \mapsto -\log(x).$$

□

Theorem 71. For all $n \geq 2$ the following map is an isomorphism of topological spaces:

$$a_n \colon \overline{\mathcal{M}}_{0,n+1} \to \mathrm{W(Comm)}(n) \colon [(G,L,f,l)] \mapsto [(G,L,f,\alpha \circ l)].$$

Furthermore, there is a structure of symmetric non-unital topological operad on the collection $\{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq 2}$, with composition given by the tropical clutching map, and we denote this operad by $\boldsymbol{\overline{\mathcal{M}}_0}$. The morphisms α_n extend to an isomorphism of non-unital topological operads from $\overline{\mathcal{M}}_0$ to the operad obtained from W(Comm) by forgetting terms of arity 1 and 0.

Proof. By Lemma 70 we know that there is an isomorphism of topological monoids $\alpha \colon ([0,\infty],+) \to ([0,1],\star)$. For all $n \geq 2$ the map

$$a_n \colon \overline{\mathcal{M}}_{0,n+1} \to \mathrm{W(Comm)}(n) : (G,L,f,l) \mapsto [(G,L,f,\alpha \circ l)]$$

is well-defined, since the data $(G,L,f,\alpha \circ l)$ gives an n-tree with edges labelled by numbers in $[0,1]$ so that the external edges are labelled by 1. This map is continuous since α is continuous. On the other hand, every operation $[(T,h)]$ in $\mathrm{W(Comm)}(n)$ has a representant $(\widetilde{T},\widetilde{h})$ with none of the edges labelled by zero and which is unique up to a change of the planar structure, so we can define the continuous map

[4] The proof of this Lemma was suggested by Todd Trimble in a blog comment.

$$a_n^{-1}\colon [(T,l)] \mapsto (\widetilde{T}, \alpha^{-1} \circ \widetilde{h})$$

which gives an inverse to a_n.

Next we show tha the collection $\{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq 2}$ has the structure of a symmetric non-unital topological operad. The action of the symmetric groups is given by

$$(C, x_0, x_1, x_2 \ldots, x_{n+1}) \mapsto (C, x_0, x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}), \text{ for } \sigma \in \Sigma_n.$$

Further, the tropical clutching map is easily seen to be well-defined and continuous and thus this data gives an operad structure on $\{\overline{\mathcal{M}}_0(n)\}_{n\geq 2} := \{\overline{\mathcal{M}}_{0,n+1}\}_{n\geq 2}$.

Since α is a homomorphism, the collection of morphisms $\{\alpha_n\}_{n\geq 2}$ extends to a morphism of topological operads from $\overline{\mathcal{M}}_0$ to the operad obtained from W(Comm) by forgetting terms of arity 1 and 0. □

Looking at the picture of a tropical moduli space, one could object that we drew them as if they were unrooted trees, not rooted trees. Indeed $\overline{\mathcal{M}}_0$ is more than an operad, it is a cyclic operad. This is a further analogy with the classical case, see for example [**Mar06**, Section 6].

Definition 72. [**Mar06**, Proposition 42] A non-unital **cyclic operad** is a non-unital operad $\mathcal{P}$ together with for each $n \geq 0$ a right action of Σ_n^+ such that for each $1 \leq i \leq m$ and for each $n \geq 0$, for all $p \in \mathcal{P}(m)$ and for all $q \in \mathcal{P}(n)$ the following is satisfied:

$$(p \circ_i q)\tau_{m+n-1} = \begin{cases} (q\tau_n) \circ_n (p\tau_m), \text{ if } i = 1, \\ (p\tau_m) \circ_{i-1} q, \text{ for } 2 \leq i \leq m \end{cases}$$

where $\boldsymbol{\Sigma_n^+}$ denotes the permutation group of the set $\{0, 1, \ldots, n\}$ and $\boldsymbol{\tau_n}$ denotes the permutation $(0 \ldots n) \in \Sigma_n^+$.

Theorem 73. The operad $\overline{\mathcal{M}}_0$ is a cyclic operad.

Corollary 74. If we forget the terms of arity 1 and 0, the operad W(Comm) is a cyclic operad.

APPENDIX A

Closed monoidal categories, enrichment and the endomorphism operad

The aim of this appendix is to explain how a closed monoidal category is enriched in itself. This in turn makes it possible to define endomorphism operads, which permit to give a characterization of an algebra X over an operad as a morphism from the operad to the endomorphism operad of X.

1. Every closed monoidal category is enriched over itself

Definition 75. A symmetric monoidal category $\mathcal{C}$ is **closed** if there is a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$, called **internal Hom functor** such that for all $X \in \mathcal{C}$ the functor $- \otimes C$ has a left adjoint $F(C, -)$.

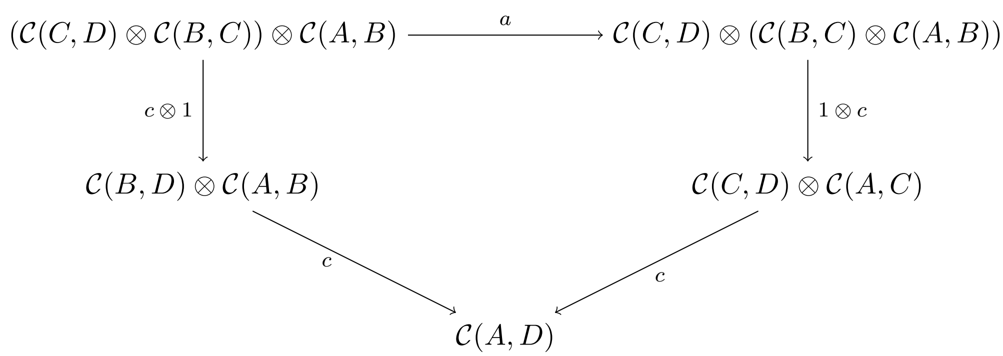
Remark 76. If $\mathcal{C}$ is not symmetric, then $- \otimes C$ and $C \otimes -$ are two different functors and may or may not have a right adjoint.

Definition 77. Let $\mathcal{V} = (\mathcal{V}_0, \otimes, K, a, l, r)$ be a monoidal category. A **category $\mathcal{C}$ enriched over $\mathcal{V}$** (or **$\mathcal{V}$-category**) consists of the following data:

- A set ob$\mathcal{C}$ whose elements are called **objects of $\mathcal{C}$**
- For each pair A, B of objects of $\mathcal{C}$ an object $\mathcal{C}(A, B) \in \mathcal{V}_0$
- For every triple of objects A, B, C in $\mathcal{C}$ a morphism $c_{ABC} : \mathcal{C}(B, C) \otimes \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ in $\mathcal{V}_0$
- For every object A in $\mathcal{C}$ a morphism $\phi_A : K \to \mathcal{C}(A, A)$ in $\mathcal{V}_0$.

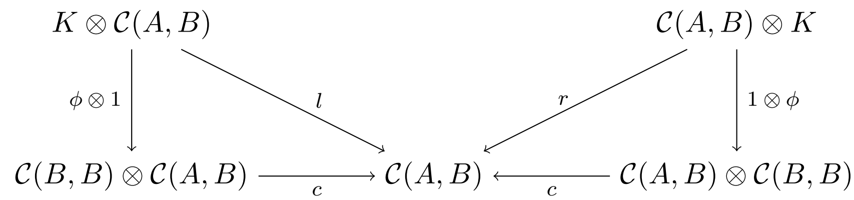
This data satisfies the commutativity of the following diagrams:

- **Associativity**



where for $C \in \mathcal{V}_0$ the arrow $1 : C \to C$ denotes the identity morphism in $\mathrm{Hom}_{\mathcal{V}_0}(C, C)$

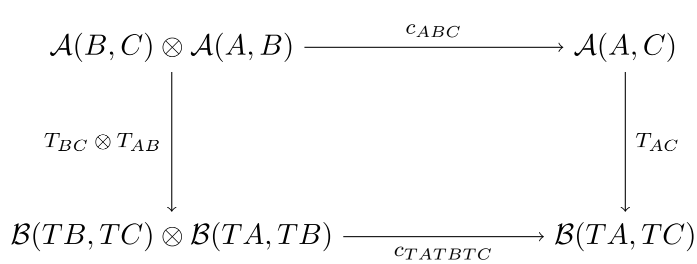
- **Unit**

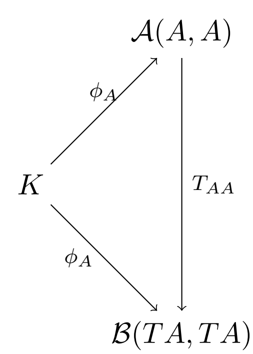


Definition 78. Let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{V}$-categories. A **$\mathcal{V}$-functor $\mathbf{T} : \mathcal{A} \to \mathcal{B}$** is given by the following data

- A map $T : \mathrm{ob}\mathcal{A} \to \mathrm{ob}\mathcal{B}$
- For each pair $A, B \in \mathrm{ob}\mathcal{A}$ a morphism $T_{A,B} : \mathcal{A}(A, B) \to \mathcal{B}(TA, TB)$ in $\mathcal{V}$.

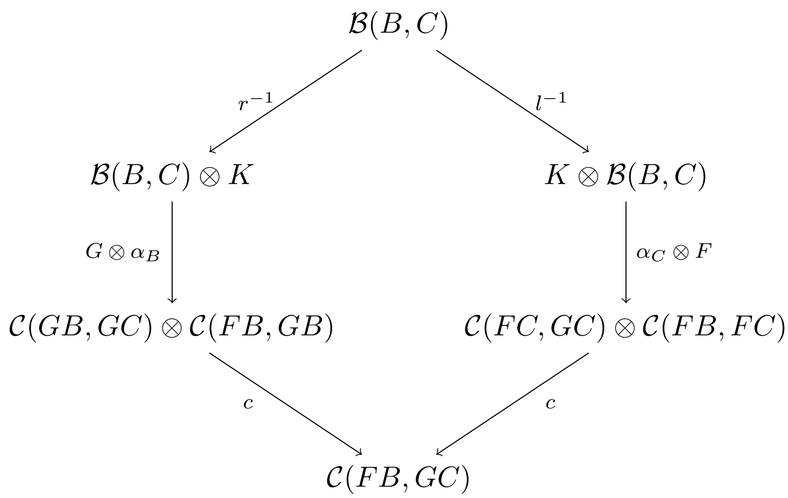
This data satisfies the commutativity of the following diagrams:





Definition 79. Given two $\mathcal{V}$-functors $\mathcal{B} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{C}$, a **$\mathcal{V}$-natural transformation** between them consists of a collection of morphisms $\{\alpha_A : K \to \mathcal{C}(FA, GA)\}_{A \in \mathrm{ob}\mathcal{A}}$ in $\mathcal{V}_0$ which

satisfy the commutativity of the following diagram:



Given $\mathcal{V}$-functors F, G, H from $\mathcal{B}$ to $\mathcal{C}$ and $\mathcal{V}$-natural transformations α from F to G and β from G to H, for any object $B \in \mathrm{ob}\mathcal{B}$ we can define the **composite** $(\beta \cdot \alpha)_B$ as

$$K \to K \otimes K \xrightarrow{\beta_B \otimes \alpha_B} \mathcal{C}(GB, HB) \otimes \mathcal{C}(FB, GB) \xrightarrow{c} \mathcal{C}(FB, HB)$$

This is a $\mathcal{V}$-natural transformation from F to H: the commutativity of the diagram in Definition 79 is given by the naturality of α and β and by the symmetry and associativity in the monoidal category $\mathcal{V}$.

Let $\mathcal{I}$ denote the $\mathcal{V}$-category with only one object X and with $\mathcal{I}(X, X) = K$ and with $\phi_X : K \to \mathcal{I}(X, X)$ the identity morphism of K (the commutativity of the diagram for unit in Definition 77 is trivial, if one considers that $l_K = r_K$ in a monoidal category [**JS93**, Proposition 1.1]). Remark that given a $\mathcal{V}$-category $\mathcal{A}$, a functor from $\mathcal{I}$ to $\mathcal{A}$ ca be identified with an object of $\mathcal{A}$, and a natural transformation between two such functors $F(X) = A$ and $G(X) = B$ for $A, B \in \mathrm{ob}\mathcal{A}$, with a morphism $K \to \mathcal{A}(A, B)$ in $\mathcal{V}_0$.

Definition 80. Let $\mathcal{A}$ be a $\mathcal{V}$-category. The **underlying category** of $\mathcal{A}$ is the (ordinary) category $\mathcal{A}_0$ with objects functors from $\mathcal{I}$ to $\mathcal{A}$ and morphisms natural transformations between such functors.

Thus the underlying category $\mathcal{A}_0$ of a $\mathcal{V}$-category $\mathcal{A}$ has the same objects as $\mathcal{A}$ and for every pair of objects A, B Hom-sets $Hom_{\mathcal{A}_0}(A, B) = Hom_{\mathcal{V}_0}(K, \mathcal{A}(A, B))$.

Proposition 81. Assume that $\mathcal{V}$ is a symmetric monoidal closed category. Let $F : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$ denote the internal hom functor and

$$\eta_Y : Y \to F(X, Y \otimes X)$$

$$\epsilon_Y : F(X,Y) \otimes X \to Y$$

denote the unit and counit of the adjunction $- \otimes X \dashv F(X, -)$.

There is a $\mathcal{V}$-category $\bar{\mathcal{V}}$ with objects those of $\mathcal{V}_0$ and whose hom-objects are given by $F(X,Y) \in \mathcal{V}_0$ for every pair of objects $X, Y \in \mathcal{V}_0$ and such that the underlying category of $\bar{\mathcal{V}}$ is isomorphic to $\mathcal{V}_0$.

Proof. The composition $c : F(Y,Z) \otimes F(X,Y) \to F(X,Z)$ is given by

$$(F(Y,Z) \otimes F(X,Y)) \otimes X \xrightarrow{a} F(Y,Z) \otimes (F(X,Y) \otimes X) \xrightarrow{1 \otimes \epsilon} F(Y,Z) \otimes Y \xrightarrow{\epsilon} Z$$

under the adjunction.

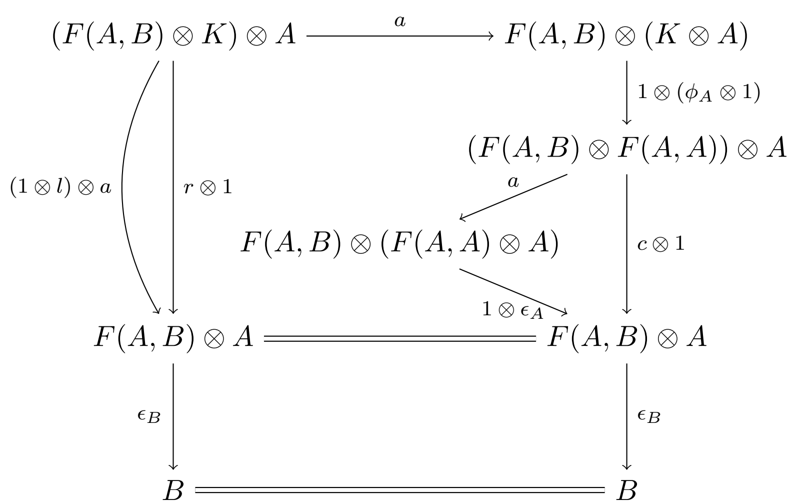
The unit $\phi_A : K \to F(A,A)$ is given by

$$l_A : K \otimes A \to A$$

under the adjunction.

This data satisfies the commutativity of the associativity and unit diagrams in Definition 77: consider the following two diagrams

$$\begin{array}{ccc}
((F(C,D) \otimes F(B,C)) \otimes (F(A,B)) \otimes A & \xrightarrow{a \otimes 1} & (F(C,D) \otimes (F(B,C) \otimes F(A,B))) \otimes A \\
\Big\downarrow{\scriptstyle a} & & \Big\downarrow{\scriptstyle a} \\
F(C,D) \otimes (F(B,C) \otimes (F(A,B) \otimes A)) & \xleftarrow{a} & F(C,D) \otimes ((F(B,C) \otimes F(A,B)) \otimes A) \\
\Big\downarrow{\scriptstyle 1 \otimes (1 \otimes \epsilon)} & & \Big\downarrow{\scriptstyle 1 \otimes (c \otimes 1)} \\
F(C,D) \otimes (F(B,C) \otimes B) & & F(C,D) \otimes ((F(A,C) \otimes A) \\
\Big\downarrow{\scriptstyle 1 \otimes \epsilon} & & \Big\downarrow{\scriptstyle 1 \otimes \epsilon} \\
F(C,D) \otimes C & = & F(C,D) \otimes C \\
\Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow{\scriptstyle \epsilon} \\
D & = & D
\end{array}$$



In the first diagram, the first square commutes by the associativity in the monoidal category; the second square commutes by the relation of the counit ϵ to the natural bijection $\pi_{Y,Z} : \mathcal{V}(Y \otimes X, Z) \to \mathcal{V}(Y, F(X, Z))$ given by the adjunction: for any $f : B \to F(X, A)$, we have

$$\pi^{-1}_{B,A} f = \epsilon_A \circ f \otimes X : B \otimes X \to A,$$

see [**Lan98**, Theorem 1, (ii) page 82]. Hence $\epsilon(1 \otimes \epsilon)a \overset{def}{=} \pi^{-1}(c) = \epsilon(c \otimes 1)$.

In the second diagram, the two different paths on the left commute by the axiom for unit in the monoidal category; the triangle commutes since $\epsilon(1 \otimes \epsilon)a = \epsilon(c \otimes 1)$ and ϵ is invertible. Hence the first square commutes since $\epsilon_A(\phi_A \otimes 1) = l_A$, by the relation of the conuit ϵ to π and definition of ϕ_A. The commutativity of the other triangle for unit required in Definition 77 is proven in a similar way.

The underlying category of $\bar{\mathcal{V}}$ is isomorphic to $\mathcal{V}_0$:
the natural isomorphisms $\pi_{Y,Z} : \mathcal{V}_0(X \otimes Y, Z) \cong \mathcal{V}_0(X, F(Y, Z))$ give natural bijections $\psi_{Y,Z} : \mathcal{V}_0(Y, Z) \cong \mathcal{V}_0(K, F(Y, Z))$, since $\mathcal{V}_0(K, -) : \mathcal{V}_0 \to \mathsf{Set}$. We thus have an isomorphism between the underlying category of $\bar{\mathcal{V}}$ and $\mathcal{V}_0$. □

We thus have proved that any monoidal symmetric closed category is enriched over itself.

In analogy to the product of ordinary categories, if $\mathcal{V}$ is symmetric we can define the product of two $\mathcal{V}$-categories:

Definition 82. Given two $\mathcal{V}$-categories $\mathcal{A}$ and $\mathcal{B}$, we can define a $\mathcal{V}$-category $\mathcal{A} \otimes \mathcal{B}$, called the **product** of $\mathcal{A}$ and $\mathcal{B}$. It consists of the following data:

- the set of objects is the cartesian product ob$\mathcal{A} \times$ ob$\mathcal{B}$
- for each pair of objects $(A, B), (A', B')$ of $\mathcal{A} \otimes \mathcal{B}$ an object $\mathcal{A}(A, A') \otimes \mathcal{B}(B, B') \in \mathcal{V}_0$

- for each triple of objects $(A,B),(A',B'),(A'',B'')$ of $\mathcal{A}\otimes\mathcal{B}$ a morphism

$$(\mathcal{A}(A',A'')\otimes\mathcal{B}(B',B''),\mathcal{A}(A,A')\otimes\mathcal{B}(B,B'))\to\mathcal{A}(A,A'')\otimes\mathcal{B}(B,B'')$$

  in $\mathcal{V}_0$ given by $c_{AA'A''}\otimes c_{BB'B''}$
- for each object (A,B) of $\mathcal{A}\otimes\mathcal{B}$ a morphism $K\to\mathcal{A}(A,A)\otimes\mathcal{B}(B,B))$ in $\mathcal{V}_0$, given by $\phi_A\otimes\phi_B\circ l_K^{-1}$ (where $l_K : K\otimes K\to K$ is the left unity constraint of the monoidal category and by [Proposition 1.1 page 23][**JS93**] we can without restriction of generality choose the left over the right constraint, since $l_K = r_K$).

This data is easily seen to satisfy the commutativity of the associativity and unit diagrams in Definition 77.

Now we define a $\mathcal{V}$-functor $\mathbf{Ten} : \bar{\mathcal{V}}\otimes\bar{\mathcal{V}}\to\bar{\mathcal{V}}$ given by the data

- $Ten : \mathrm{ob}\bar{\mathcal{V}}\times\mathrm{ob}\bar{\mathcal{V}}\to\mathrm{ob}\bar{\mathcal{V}} : (X,Y)\mapsto X\otimes Y$
- $Ten_{(X,Y),(X',Y')} : F(X,X')\otimes F(Y,Y')\to F(X\otimes X',Y\otimes Y')$, for each pair of objects $(X,Y),(X',Y')$ of $\bar{\mathcal{V}}\otimes\bar{\mathcal{V}}$, which is given by

$$(F(X,X')\otimes F(Y,Y'))\otimes(X\otimes Y)\xrightarrow{m}(F(X,X')\otimes X)\otimes(F(Y,Y')\otimes Y)\xrightarrow{\epsilon\otimes\epsilon}X'\otimes Y'$$

  under the adjunction, where the morphism m is given by the obvious iteration and composition of the braiding and the associativity constraint.

This data satisfies the axioms of Definition 78, see also [**Kel05**, page 16].
The above definition of product can be extended to n $\mathcal{V}$-categories for n a natural number, and, similarly, the definition of the $\mathcal{V}$-functor Ten can be extended to a functor $\underbrace{\bar{\mathcal{V}}\otimes\cdots\otimes\bar{\mathcal{V}}}_{n\text{ factors}}\to\bar{\mathcal{V}}$. We will simply call this functor Ten.

2. Endomorphism operad and algebras over an operad

Now suppose that $\mathcal{C}$ is a symmetric monoidal category which is closed. We adopt the same notation as for Proposition 81 for the internal Hom functor and the unit and counit of the adjunction. In the following we construct for every object X of $\mathcal{C}$ an operad $\mathbf{End}_{\boldsymbol{X}}=\{\mathrm{End}_X(n)\}_{n\in\mathbb{Z}_{\geq 0}}$ in $\mathcal{C}$, which is called the **endomorphism operad of *X***.
We write X^n for the n-fold $\otimes$-product of X.
Define $\mathrm{End}_X(n) = F(X^n,X)$ and

$$\gamma_{n;m_1,\dots,m_n} : \mathrm{End}_X(n)\otimes(\mathrm{End}_X(m_1)\otimes\cdots\otimes\mathrm{End}_X(m_n))\to\mathrm{End}_X(m_1+\cdots+m_n)$$

to be the composition

$$\begin{array}{c} F(X^n, X) \otimes F(X^{m_1}, X) \otimes \cdots \otimes F(X^{m_n}, X) \\ \Big\downarrow {\scriptstyle 1 \otimes Ten} \\ F(X^n, X) \otimes F(X^{m_1 + \cdots + m_n}, X^n) \\ \Big\downarrow {\scriptstyle c} \\ F(X^{m_1 + \cdots + m_n}, X) \end{array}$$

where the morphism c is the composition given by the enrichment of $\mathcal{C}$ over itself.

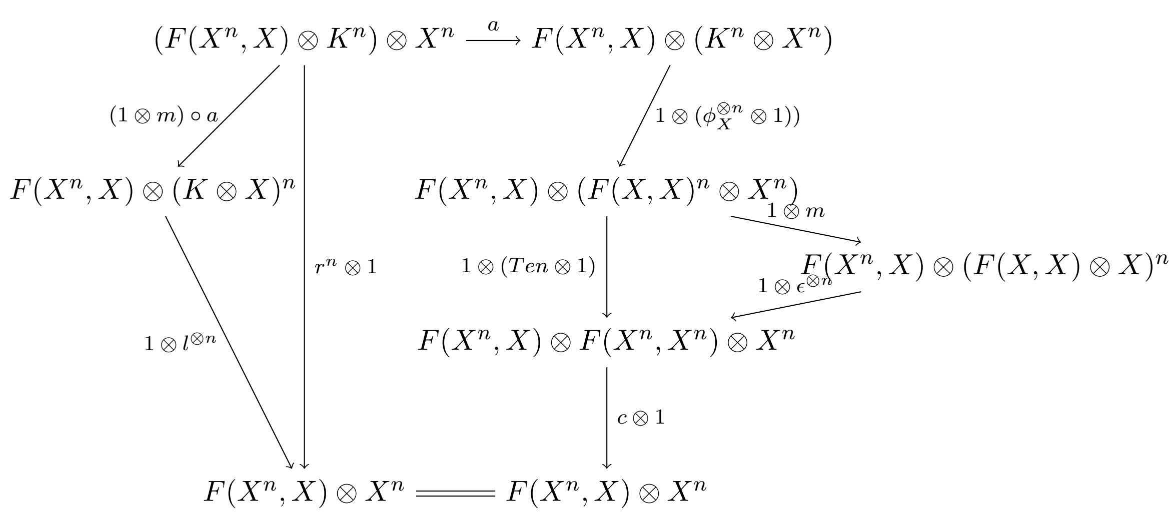
There is a left action of Σ_n on X^n given by the permutation of the factors in the product:

$$\Sigma_n \to Aut(X^n) : \sigma \mapsto (X^n \to X_{\sigma^{-1}(1)} \otimes \cdots \otimes X_{\sigma^{-1}(n)}).$$

This left action induces a right action of Σ_n on $\mathrm{End}_X(n)$, since the functor $F(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$ is contravariant in the first argument.

Let the unit morphism $\eta : K \to \mathrm{End}_X(1)$ be given by ϕ_X.

The verification of the commutativity of the axioms in Definition 10 involves the adjunction and the axioms for a symmetric monoidal category in a similar way as done before in the proof of Proposition 81. We give here only the proof for the first triangle for unit of Definition 10. Consider the following diagram:



where for objects $A, B \in \mathcal{C}$ the morphism $m : A^n \otimes B^n \to (A \otimes B)^n$ is given by the obvious iteration and composition of the braiding and the associativity constraint.

The triangle on the left commutes by the Coherence Theorem for monoidal categories, see also [**Lan63**]. The triangle on the right commutes by definition of Ten. Then the square commutes, i.e.

$$(1 \otimes \epsilon^{\otimes n}) \circ (1 \otimes m) \circ (1 \otimes (\phi_X^n \otimes 1)) \circ a = (1 \otimes l^{\otimes n}) \circ (1 \otimes m) \circ a$$

since $\epsilon(\phi_X \otimes 1) = l_X$ by definition of ϕ_X.

Lemma 83. If the category $\mathcal{C}$ is closed a $\mathcal{P}$-algebra structure on X is equivalent to a morphism of operads $\mathcal{P} \to \mathrm{End}_X$.

Bibliography

[ACP12] D. Abramovich, L. Caporaso, and S. Payne. The tropicalization of the moduli space of curves. *ArXiv e-prints*, December 2012.

[BHV01] Louis J. Billera, Susan P. Holmes, and Karen Vogtmann. Geometry of the space of phylogenetic trees. *Advances in Applied Mathematics*, 27(4):733 – 767, 2001.

[BM02] C. Berger and I. Moerdijk. Axiomatic homotopy theory for operads. *ArXiv Mathematics e-prints*, June 2002.

[BM05] C. Berger and I. Moerdijk. The Boardman-Vogt resolution of operads in monoidal model categories. *ArXiv Mathematics e-prints*, February 2005.

[BO14] J. Baez and N. Otter. Operads and phylogenetic trees. 2014.

[BS13] E. Brugallé and K. Shaw. A bit of tropical geometry. *ArXiv e-prints*, November 2013.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces.* Lecture Notes in Mathematics, Vol. 347. Springer-Verlag, Berlin, 1973.

[Cap11] L. Caporaso. Algebraic and tropical curves: comparing their moduli spaces. *ArXiv e-prints*, January 2011.

[Cra95] Sjoerd E. Crans. Quillen closed model structures for sheaves. *Journal of Pure and Applied Algebra*, 101(1):35 – 57, 1995.

[Fre] B. Fresse. Homotopy of operads and Grothendieck–Teichmüller groups.

[Get94] E. Getzler. Operads and moduli spaces of genus 0 Riemann surfaces. In *eprint arXiv:alg-geom/9411004*, page 11004, November 1994.

[GK07] V. Ginzburg and M. Kapranov. Koszul duality for Operads. *ArXiv e-prints*, September 2007.

[GM05] A. Gathmann and H. Markwig. Kontsevich's formula and the WDVV equations in tropical geometry. *ArXiv Mathematics e-prints*, September 2005.

[GM07] N. Grigg and N. Manwaring. An Elementary Proof of the Fundamental Theorem of Tropical Algebra. *ArXiv e-prints*, July 2007.

[Hir09] P.S. Hirschhorn. *Model Categories and Their Localizations*. Mathematical Surveys and Monographs. Amer Mathematical Society, 2009.

[Hov07] M. Hovey. *Model Categories*. Mathematical surveys and monographs. American Mathematical Society, 2007.

[JS93] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[Kel05] G. M. Kelly. Basic concepts of enriched category theory, 2005.

[Lan63] Saunders M. Lane. Natural associativity and commutativity. *Rice University Studies*, 49:28–46, 1963.

[Lan98] Saunders M. Lane. *Categories for the Working Mathematician (Graduate Texts in Mathematics)*. Springer, 2nd edition, September 1998.

[Lei04] Tom Leinster. Coproducts of operads, and the W-construction. 2004.

[Lit05] G. L. Litvinov. The Maslov dequantization, idempotent and tropical mathematics: A brief introduction. *ArXiv Mathematics e-prints*, July 2005.

[LV12] J.L. Loday and B. Vallette. *Algebraic Operads*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete. Springer, 2012.

[Mar06] M. Markl. Operads and PROPs. *ArXiv Mathematics e-prints*, January 2006.
[May97] J. P. May. Definitions: operads, algebras and modules. In *Contemporary Mathematics 202*, pages 1–7, 1997.
[Mik03] G. Mikhalkin. Enumerative tropical algebraic geometry in $\mathbb{R}^2$. *ArXiv Mathematics e-prints*, December 2003.
[Mik06] G. Mikhalkin. Tropical Geometry and its applications. *ArXiv Mathematics e-prints*, January 2006.
[Mik07] G. Mikhalkin. Moduli spaces of rational tropical curves. *ArXiv e-prints*, April 2007.
[MS12] D. McDuff and D. Salamon. *J-holomorphic Curves and Symplectic Topology*. American Mathematical Society colloquium publications. American Mathematical Society, 2012.
[MT11] F. Muro and A. Tonks. Unital associahedra. *ArXiv e-prints*, October 2011.
[MZ06] G. Mikhalkin and I. Zharkov. Tropical curves, their Jacobians and Theta functions. *ArXiv Mathematics e-prints*, December 2006.
[Pin98] Jean-Eric Pin. Tropical semirings. *Idempotency*, 1998.
[Qui67] D.G. Quillen. *Homotopical Algebra*. Lecture Notes in Mathematics. Springer, 1967.
[Sim88] Imre Simon. Recognizable sets with multiplicities in the tropical semiring, 1988.
[Sm72] Arne Strø m. The homotopy category is a homotopy category. *Archiv der Mathematik*, 23(1):435–441, 1972.
[SS98] S. Schwede and B. E. Shipley. Algebras and modules in monoidal model categories. *ArXiv Mathematics e-prints*, January 1998.
[SS04] D. Speyer and B. Sturmfels. Tropical Mathematics. *ArXiv Mathematics e-prints*, August 2004.
[Ste67] N. E. Steenrod. A convenient category of topological spaces. *The Michigan Mathematical Journal*, 14(2):133–152, 05 1967.
[Str09] Neil Strickland. The category of CGWH spaces. `http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf`, 2009.
[Str11] J. Strom. *Modern Classical Homotopy Theory*. Graduate studies in mathematics. American Mathematical Society, 2011.
[Vog03] R.M. Vogt. Cofibrant operads and universal E_∞ operads. *Topology and its Applications*, 133(1):69 – 87, 2003.
[Wyl73] Oswald Wyler. Convenient categories for topology. *General Topology and its Applications*, 3(3):225 – 242, 1973.