Stratifying multi-parameter persistent homology

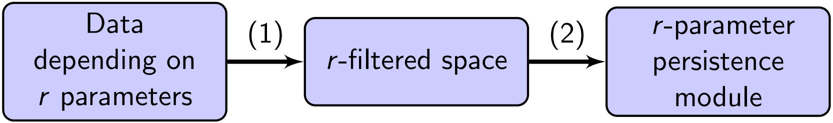
Nina Otter

Mathematical Institute, University of Oxford
The Alan Turing Institute

Applied Algebraic Topology 2017
Sapporo, 9 August 2017

joint with Hal Schenck, Heather Harrington and Ulrike Tillmann

Persistent homology pipeline



Step (1): from data to multi-filtered spaces

Define the following partial order on $\mathbb{N}^r$:
$(u_1, \ldots, u_r) \preccurlyeq (v_1, \ldots, v_r)$ iff $u_i \leq v_i$ for all $i = 1, \ldots, r$.

A **multi-filtered space** K is a tuple $(K, \{K_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}^r})$ with $K_{\mathbf{u}} \subseteq K_{\mathbf{v}}$ whenever $\mathbf{u} \preccurlyeq \mathbf{v}$ in $\mathbb{N}^r$ and $K = \cup_{\mathbf{u} \in \mathbb{N}^r} K_{\mathbf{u}}$.

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digital image with color vectors of length r	$\longrightarrow$	r-filtered cubical complex

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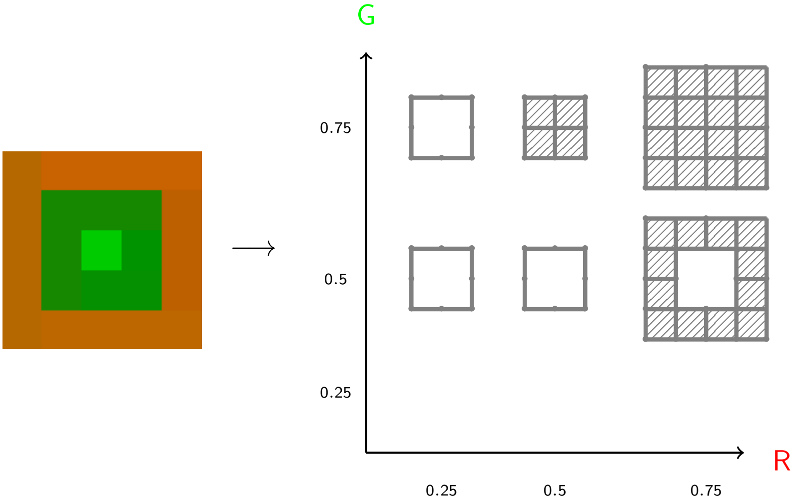
Step (1): from data to multi-filtered spaces: example



Step (2): from multi-filtered spaces to multi-parameter persistence modules

r-filtered space $\xrightarrow{H_i}$ *r*-parameter persistence module

An *r*-**parameter persistence module** is a tuple $(\{M_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^r}, \{\phi_{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\preccurlyeq\mathbf{v}\in\mathbb{N}^r})$ where:

- for each $\mathbf{u} \in \mathbb{N}^r$ we have that $M_{\mathbf{u}}$ is a $\mathbb{K}$-vector space
- for every $\mathbf{u} \preccurlyeq \mathbf{v}$ we have that $\phi_{\mathbf{u},\mathbf{v}} \colon M_{\mathbf{u}} \to M_{\mathbf{v}}$ is a $\mathbb{K}$-linear map such that whenever $\mathbf{u} \preccurlyeq \mathbf{u}' \preccurlyeq \mathbf{u}''$ we have

$$\phi_{\mathbf{u}',\mathbf{u}''} \circ \phi_{\mathbf{u},\mathbf{u}'} = \phi_{\mathbf{u},\mathbf{u}''}.$$

In other words, an *r*-parameter persistence module is a functor $F \colon \mathbb{N}^r \to \mathbb{K}\mathrm{Vect}$.

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Put an $\mathbb{N}^r$-grading on the ring $\mathbb{K}[x_1, \ldots, x_r]$:

$$\mathbb{K}[x_1, \ldots, x_r] = \bigoplus_{\mathbf{u} \in \mathbb{N}^r} \mathbb{K}\mathbf{x}^{\mathbf{u}},$$

where $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \ldots x_r^{u_r}$, so every variable x_i has degree $\mathbf{e_i} \in \mathbb{N}^r$.

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Theorem (Carlsson, Zomorodian, 2009[1])

There is an isomorphism of categories between the category of r-parameter persistence modules and the category of $\mathbb{N}^r$-graded modules over $\mathbb{K}[x_1, \ldots, x_r]$.

[1] G. Carlsson, A. Zomorodian *The theory of multidimensional persistence*, Discrete & Computational Geometry, 2009

Structure theorem for f.g. graded modules over a PID

Theorem (Webb 1985[2])

For any finitely generated $\mathbb{N}$*-graded module* M *over* $\mathbb{K}[x]$

$$M \cong \left(\bigoplus_{i=1}^{n} x^{\alpha_i} \mathbb{K}[x]) \right) \oplus \left(\bigoplus_{j=1}^{m} x^{\beta_j} \mathbb{K}[x] / x^{\beta_j + \gamma_j} \right) .$$

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This gives:

- n infinite intervals $[\alpha_i, \infty)$ for $i = 1, \ldots, r$
- m finite intervals $[\beta_j, \beta_j + \gamma_j)$ for $j = 1, \ldots, m$.

This collection of intervals is called **barcode**, and it is a complete invariant for 1-parameter persistence modules.

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Problem: For $r > 1$ there is no analogous decomposition.

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Therefore, studying the homology of r-filtered spaces amounts to studying graded modules over $\mathbb{K}[x_1, \dots, x_r]$.

Desiderata for invariants for applications

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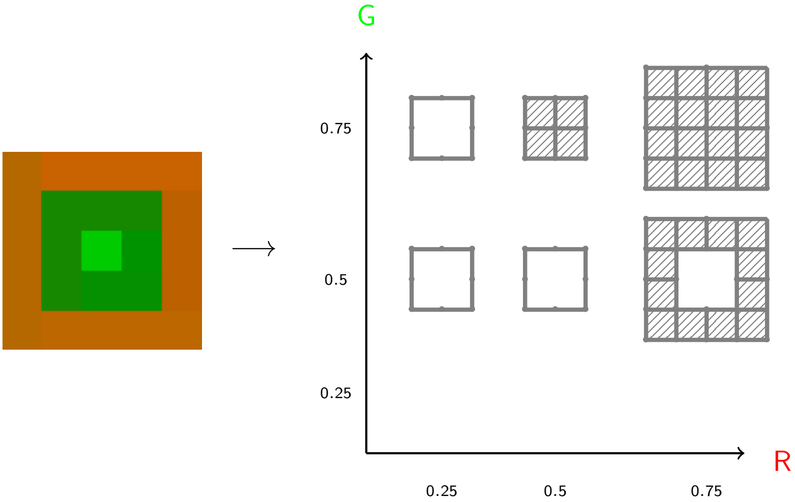
From (Carlsson, Zomorodian 2009):

> *Our study of multigraded objects shows that no complete discrete invariant exists for multidimensional persistence. We still desire a discriminating invariant that captures persistent information, that is, homology classes with large persistence.*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Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

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Definition

For such a prime $\mathfrak{p}$ define the **complement support**

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The **support shape** of M is $ss(M) = \bigcup_{\mathfrak{p} \in Ass(M)} c_{\mathfrak{p}}$.

Stratification of the support shape

Given a sequence $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ of associated primes of M, one obtains a nested sequence

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Note: $ss(M)$ is completely determined by the minimal associated primes.

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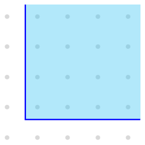

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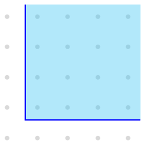

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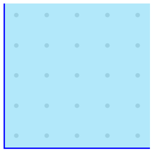

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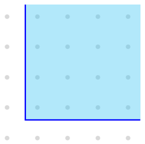

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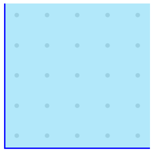

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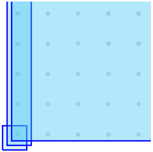

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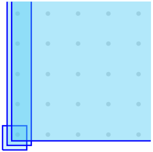

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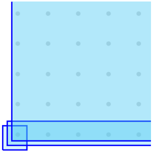

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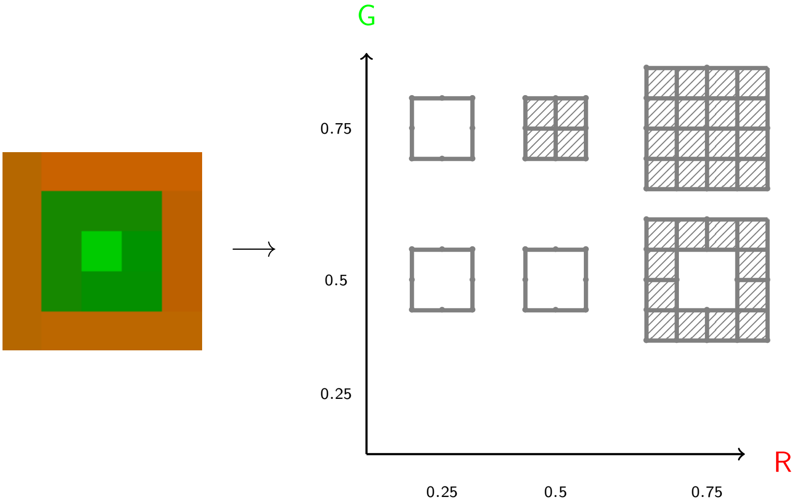
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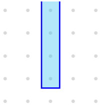
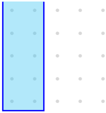
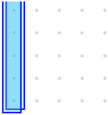
Information forgotten by $c_{\mathfrak{p}}$ and $ss(M)$

module	points at which the module does not vanish	information forgotten
$M_1 = \dfrac{S(-1,-2)}{x_1}$		"translation"
$M_2 = \dfrac{S}{x_1^2}$		"thickness"
$M_3 = \dfrac{S}{x_1} \oplus \dfrac{S}{x_1}$		"multiplicity"

Generalisation of definition of birth and death

We say that a homogeneous element a of M is:

- **born at** $(u_1, \ldots, u_r) \in \mathbb{N}^r$ if the degree of a is $(u_1, \ldots, u_r)$ and a is not in the image of any sum of maps $\sum_{\mathbf{v}} x^{\mathbf{v}}$ for any $\mathbf{v} \prec \mathbf{u}$.
- If $Ann(a) \neq (0)$ let $D \subset \mathbb{N}^r$ be the subset of $\mathbb{N}^r$ obtained from the set of degrees of the set of minimal generators of $Ann(a)$ by adding to each degree the degree of a. Then we say that a **dies in degrees** D.
- If $Ann(a) = (0)$ we say that a **lives forever**.
- If $\sqrt{Ann(a)} = \langle x_{i_1}, \ldots, x_{i_k} \rangle = \mathfrak{p}$, we say that a **lives along** $c_{\mathfrak{p}} \subset \mathbb{N}^r$. In this case we say that a has **support dimension** $r - k$.

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We call elements of support dimension 0 **transient components**, elements of support dimension $1 \leq d < r$ **persistent components**, and elements of support dimension r **fully persistent components**.

Rank and local cohomology

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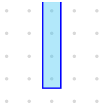
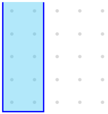
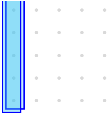
  - $H^0_{\mathfrak{p}}(M)$ is finitely generated as an $\mathbb{K}[x_{k+1}, \ldots, x_r]$-module.

Example

module	points at which the module does not vanish	
$M_1 = \dfrac{S(-1,-2)}{x_1}$		$\mathrm{rk}_{\mathbb{K}[x_2]} H^0_{x_1}(M_1) = 1$
$M_2 = \dfrac{S}{x_1^2}$		$\mathrm{rk}_{\mathbb{K}[x_2]} H^0_{x_1}(M_2) = 2$
$M_3 = \dfrac{S}{x_1} \oplus \dfrac{S}{x_1}$		$\mathrm{rk}_{\mathbb{K}[x_2]} H^0_{x_1}(M_3) = 2$

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- The stability of the stratification probably depends on the codimension of the associated primes.
- Associated primes do not give information about non-trivial second syzygies (nor higher ones).