Stratifying multiparameter persistent homology

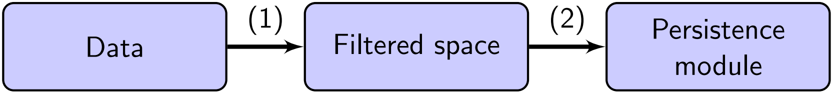
Nina Otter

Mathematical Institute, University of Oxford
The Alan Turing Institute

joint with Hal Schenck, Heather Harrington, and Ulrike Tillmann

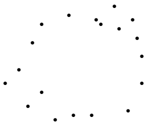
Preprint available at https://arxiv.org/abs/1708.07390

Persistent homology pipeline

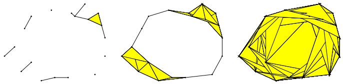


Step (1): from data to filtered spaces

finite metric space



filtered simplicial complex



Step (2): from filtered spaces to persistence modules

filtered simplicial complex

persistence module

$K_{\epsilon_1} \subseteq \cdots \subseteq K_{\epsilon_n} = K$

for $\epsilon_1 \leq \cdots \leq \epsilon_n$

$H_p(K_{\epsilon_1}) \xrightarrow{f_{1,2}} \dots \xrightarrow{f_{n-1,n}} H_p(K_{\epsilon_n})$

Step (2): from filtered spaces to persistence modules

filtered simplicial complex

$K_{\epsilon_1} \subseteq \cdots \subseteq K_{\epsilon_n} = K$
for $\epsilon_1 \leq \cdots \leq \epsilon_n$

persistence module

$$H_p(K_{\epsilon_1}) \xrightarrow{f_{1,2}} \ldots \xrightarrow{f_{n-1,n}} H_p(K_{\epsilon_n})$$

More precisely, we obtain a tuple $(\{H_p(K_{\epsilon_i})\}_{i=1}^n, \{f_{i,j}\}_{i \leq j})$ such that $f_{k,j} \circ f_{i,k} = f_{i,j}$ for all $i \leq k \leq j$.

This is the p**th persistent homology** of $(K, \{K_{\epsilon_i}\}_{i=1}^n)$.

Persistence modules

In general,

- a sequence $\{M_i\}_{i \in \mathbb{N}}$ of $\mathbb{K}$-vector spaces
- a collection $\{f_{i,j} \colon M_i \longrightarrow M_j\}_{i \leq j}$ of linear maps such that $f_{k,j} \circ f_{i,k} = f_{i,j}$ for all $i \leq k \leq j$

is called a **persistence module**.

Persistence modules

In general,

- a sequence $\{M_i\}_{i\in\mathbb{N}}$ of $\mathbb{K}$-vector spaces
- a collection $\{f_{i,j}\colon M_i \longrightarrow M_j\}_{i\leq j}$ of linear maps such that $f_{k,j} \circ f_{i,k} = f_{i,j}$ for all $i \leq k \leq j$

is called a **persistence module**.

In other words, a persistence module is a functor $\mathbb{N} \to \mathbb{K}\mathrm{Vect}$.

Persistence modules

In general,

- a sequence $\{M_i\}_{i\in\mathbb{N}}$ of $\mathbb{K}$-vector spaces
- a collection $\{f_{i,j}\colon M_i \longrightarrow M_j\}_{i\leq j}$ of linear maps such that $f_{k,j}\circ f_{i,k} = f_{i,j}$ for all $i \leq k \leq j$

is called a **persistence module**.

In other words, a persistence module is a functor $\mathbb{N} \to \mathbb{K}\mathrm{Vect}$.

What kind of object is this?

Recall: The ring $\mathbb{K}[x]$ is $\mathbb{N}$-graded: $\mathbb{K}[x] = \bigoplus_{i\in\mathbb{N}} \mathbb{K}x^i$.

An $\mathbb{N}$-graded module M over $\mathbb{K}[x]$ is a module over $\mathbb{K}[x]$ such that $M = \bigoplus_{i\in\mathbb{N}} M_i$ and $x^j M_i \subset M_{i+j}$ for all i, j.

Correspondence theorem

$$\left(\{M_i\}_{i\in\mathbb{N}}, \{f_{i,j}\colon M_i \to M_j\}_{i\leq j}\right) \mapsto \bigoplus_{i\in\mathbb{N}} M_i \text{ with action of } x^j \text{ on } M_i$$

given by $f_{i,i+j}$

Correspondence theorem

$$\left(\{M_i\}_{i\in\mathbb{N}}, \{f_{i,j}\colon M_i \to M_j\}_{i\leq j}\right) \mapsto \bigoplus_{i\in\mathbb{N}} M_i \text{ with action of } x^j \text{ on } M_i$$

given by $f_{i,i+j}$

$$\left(\{M_i\}_{i\in\mathbb{N}}, \left\{x^{j-i}\colon M_i \to M_j\right\}_{i\leq j}\right) \leftarrow\!\shortmid M = \bigoplus_{i\in\mathbb{N}} M_i \text{ graded module}$$

Correspondence theorem

$$\left(\{M_i\}_{i\in\mathbb{N}}, \{f_{i,j}\colon M_i \to M_j\}_{i\leq j}\right) \mapsto \bigoplus_{i\in\mathbb{N}} M_i \text{ with action of } x^j \text{ on } M_i$$

given by $f_{i,i+j}$

$$\left(\{M_i\}_{i\in\mathbb{N}}, \left\{x^{j-i}\colon M_i \to M_j\right\}_{i\leq j}\right) \leftarrow M = \bigoplus_{i\in\mathbb{N}} M_i \text{ graded module}$$

Theorem (Carlsson, Zomorodian, 2005[1])

There is an isomorphism between the category of persistence modules and the category of $\mathbb{N}$-graded modules over $\mathbb{K}[x]$.

[1] G. Carlsson, A. Zomorodian *Computing persistent homology*, Discrete & Computational Geometry, 2005

Structure theorem for f.g. graded modules over a PID

Theorem (Webb 1985[2])

For any finitely generated $\mathbb{N}$-graded module M over $\mathbb{K}[x]$:

$$M \cong \left(\bigoplus_{i=1}^{n} x^{\alpha_i} \mathbb{K}[x]) \right) \oplus \left(\bigoplus_{j=1}^{m} x^{\beta_j} \mathbb{K}[x] / x^{\beta_j + \gamma_j} \right) .$$

This gives:

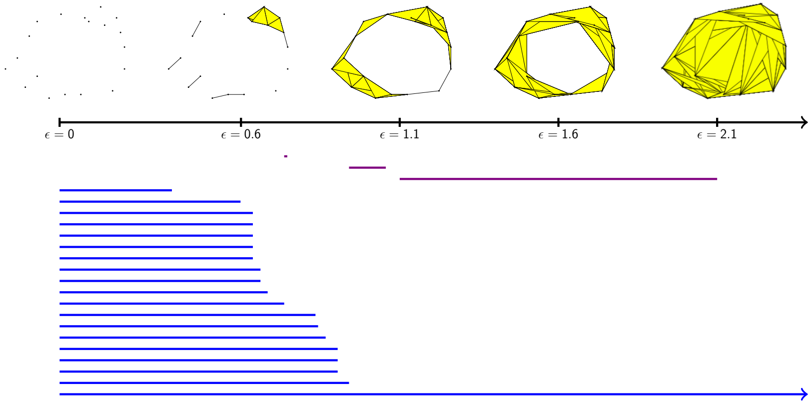
- n infinite intervals $[\alpha_i, \infty)$ for $i = 1, \dots r$
- m finite intervals $[\beta_j, \beta_j + \gamma_j)$ for $j = 1, \dots, m$.

This collection of intervals is called **barcode**, and it is a complete invariant for persistence modules.

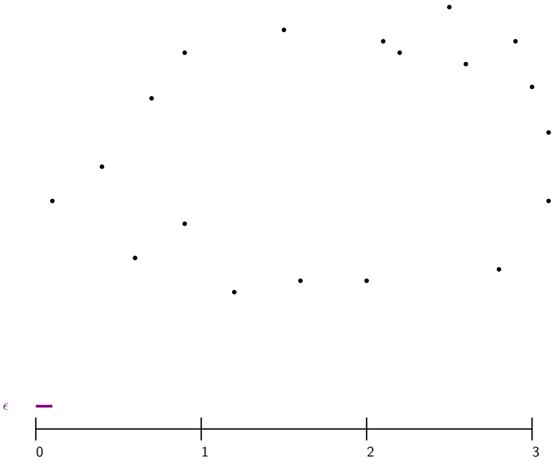
---

[2] C. Webb, *Decomposition of graded modules*, Proceedings of the AMS, 1985

Examples of barcode



Example of Barcode

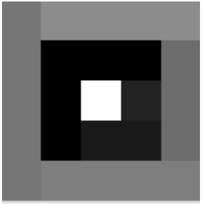


Applications of PH

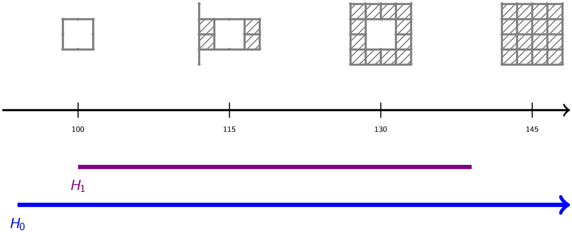
Persistent homology can be applied to, e.g.:

1. Finite metric spaces
2. Undirected weighted networks
3. Grey-scale digital images

PH to study grey-scale images



$$G = \begin{pmatrix} 115 & 119 & 119 & 119 & 119 \\ 115 & 94 & 94 & 94 & 114 \\ 115 & 94 & 139 & 100 & 114 \\ 115 & 94 & 99 & 99 & 114 \\ 115 & 117 & 117 & 117 & 117 \end{pmatrix}$$



Libraries for PH and overview of computation

- *A roadmap for the computation of persistent homology*
  N. Otter, M. Porter, U. Tillmann, P. Grindrod, H. Harrington,
  EPJ Data Science 2017 6:17 (SpringerOpen)

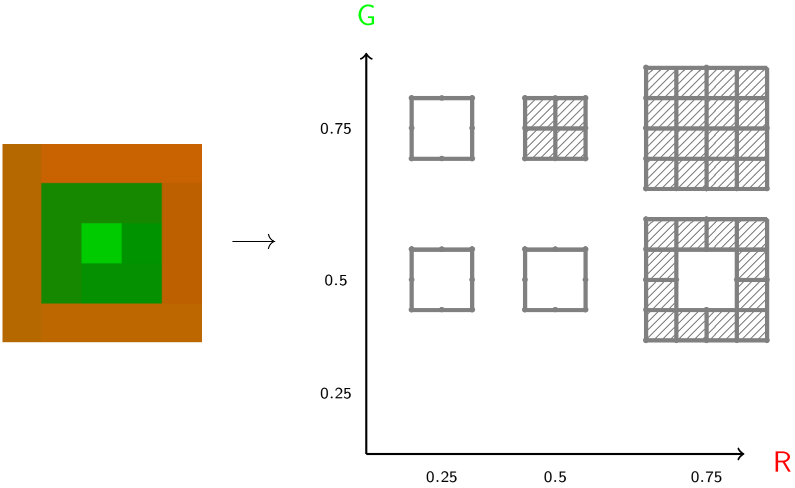
Libraries for PH and overview of computation

- *A roadmap for the computation of persistent homology*
  N. Otter, M. Porter, U. Tillmann, P. Grindrod, H. Harrington, EPJ Data Science 2017 6:17 (SpringerOpen)
- Come along to the practical session on Monday 4 September at 3pm!

Multi-parameter persistent homology

Motivation

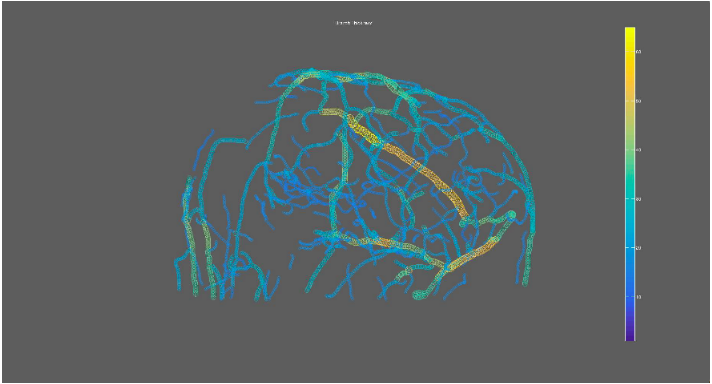
1. Data often depend on several parameters, e.g.:
   - colored digital images



Multi-parameter persistent homology

Motivation

1. Data often depend on several parameters, e.g.:
    - colored digital images
    - complex biological data sets (here: blood vessel growth in presence of tumor)



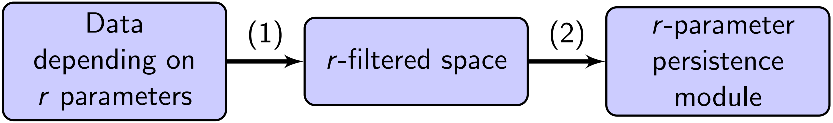
Roche, Oxford Oncology (B. Markelc), Mathematical Biology, University of Oxford (B. Stolz, H. Byrne, J. Grogan)

Multi-parameter persistent homology

Motivation

2. Outliers

Multi-parameter persistent homology pipeline



Step (1): from data to multi-filtered spaces

Define the following partial order on $\mathbb{N}^r$:
$(u_1, \ldots, u_r) \preccurlyeq (v_1, \ldots, v_r)$ iff $u_i \leq v_i$ for all $i = 1, \ldots, r$.

A **multi-filtered space** K is a tuple $(K, \{K_\mathbf{u}\}_{\mathbf{u} \in \mathbb{N}^r})$ with $K_\mathbf{u} \subseteq K_\mathbf{v}$ whenever $\mathbf{u} \preccurlyeq \mathbf{v}$ in $\mathbb{N}^r$ and $K = \cup_{\mathbf{u} \in \mathbb{N}^r} K_\mathbf{u}$.

Step (1): from data to multi-filtered spaces

Define the following partial order on $\mathbb{N}^r$:
$(u_1, \ldots, u_r) \preccurlyeq (v_1, \ldots, v_r)$ iff $u_i \leq v_i$ for all $i = 1, \ldots, r$.

A **multi-filtered space** K is a tuple $(K, \{K_{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}^r})$ with $K_{\mathbf{u}} \subseteq K_{\mathbf{v}}$ whenever $\mathbf{u} \preccurlyeq \mathbf{v}$ in $\mathbb{N}^r$ and $K = \cup_{\mathbf{u} \in \mathbb{N}^r} K_{\mathbf{u}}$.

map $f\colon X \to \mathbb{R}^r$	$\longrightarrow$	r-filtered simplicial complex
digital image with color vectors of length r	$\longrightarrow$	r-filtered cubical complex

Step (2): from multi-filtered spaces to multi-parameter persistence modules

r-filtered space $\xrightarrow{H_i}$ r-parameter persistence module

An r-**parameter persistence module** is a tuple $(\{M_{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^r}, \{\phi_{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\preccurlyeq\mathbf{v}\in\mathbb{N}^r})$ where:

- for each $\mathbf{u}\in\mathbb{N}^r$ we have that $M_{\mathbf{u}}$ is a $\mathbb{K}$-vector space
- for every $\mathbf{u}\preccurlyeq\mathbf{v}$ we have that $\phi_{\mathbf{u},\mathbf{v}}\colon M_{\mathbf{u}}\to M_{\mathbf{v}}$ is a $\mathbb{K}$-linear map such that whenever $\mathbf{u}\preccurlyeq\mathbf{u}'\preccurlyeq\mathbf{u}''$ we have

$$\phi_{\mathbf{u}',\mathbf{u}''}\circ\phi_{\mathbf{u},\mathbf{u}'}=\phi_{\mathbf{u},\mathbf{u}''}.$$

In other words, an r-parameter persistence module is a functor $F\colon\mathbb{N}^r\to\mathbb{K}\mathrm{Vect}$.

Correspondence theorem

What kind of objects are multiparameter persistence modules?

Correspondence theorem

What kind of objects are multiparameter persistence modules?

Put an $\mathbb{N}^r$-grading on the ring $\mathbb{K}[x_1, \ldots, x_r]$:

$$\mathbb{K}[x_1, \ldots, x_r] = \bigoplus_{\mathbf{u} \in \mathbb{N}^r} \mathbb{K}\mathbf{x}^{\mathbf{u}},$$

where $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \ldots x_r^{u_r}$, so every variable x_i has degree $\mathbf{e_i} \in \mathbb{N}^r$.

Correspondence theorem

What kind of objects are multiparameter persistence modules?

Put an $\mathbb{N}^r$-grading on the ring $\mathbb{K}[x_1, \ldots, x_r]$:

$$\mathbb{K}[x_1, \ldots, x_r] = \bigoplus_{\mathbf{u} \in \mathbb{N}^r} \mathbb{K}\mathbf{x}^{\mathbf{u}},$$

where $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \ldots x_r^{u_r}$, so every variable x_i has degree $\mathbf{e_i} \in \mathbb{N}^r$.

Theorem (Carlsson, Zomorodian, 2009[3])

There is an isomorphism of categories between the category of r-parameter persistence modules and the category of $\mathbb{N}^r$-graded modules over $\mathbb{K}[x_1, \ldots, x_r]$.

[3] G. Carlsson, A. Zomorodian *The theory of multidimensional persistence*, Discrete & Computational Geometry, 2009

Correspondence theorem

What kind of objects are multiparameter persistence modules?

Put an $\mathbb{N}^r$-grading on the ring $\mathbb{K}[x_1, \ldots, x_r]$:

$$\mathbb{K}[x_1, \ldots, x_r] = \bigoplus_{\mathbf{u} \in \mathbb{N}^r} \mathbb{K}\mathbf{x}^{\mathbf{u}},$$

where $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \ldots x_r^{u_r}$, so every variable x_i has degree $\mathbf{e_i} \in \mathbb{N}^r$.

Theorem (Carlsson, Zomorodian, 2009[3])

There is an isomorphism of categories between the category of r-parameter persistence modules and the category of $\mathbb{N}^r$-graded modules over $\mathbb{K}[x_1, \ldots, x_r]$.

Problem: There is no decomposition analogous to the 1-parameter case.

---

[3] G. Carlsson, A. Zomorodian *The theory of multidimensional persistence*, Discrete & Computational Geometry, 2009

Any persistence module is the homology of a filtered space

The homology of a multifiltered space is an $\mathbb{N}^r$-graded module over $\mathbb{K}[x_1, \ldots, x_r]$.

Any persistence module is the homology of a filtered space

The homology of a multifiltered space is an $\mathbb{N}^r$-graded module over $\mathbb{K}[x_1, \ldots, x_r]$.

On the other hand:

Theorem (Carlsson, Zomorodian, 2009)

For any finitely generated $\mathbb{N}^r$-graded module M there exists a finite multifiltered simplicial complex K and a positive natural number i such that M is the homology in degree i of K.

Any persistence module is the homology of a filtered space

The homology of a multifiltered space is an $\mathbb{N}^r$-graded module over $\mathbb{K}[x_1, \ldots, x_r]$.

On the other hand:

Theorem (Carlsson, Zomorodian, 2009)

For any finitely generated $\mathbb{N}^r$-graded module M there exists a finite multifiltered simplicial complex K and a positive natural number i such that M is the homology in degree i of K.

Therefore, studying the homology of r-filtered spaces amounts to studying graded modules over $\mathbb{K}[x_1, \ldots, x_r]$.

Desiderata for invariants for applications

Desiderata for invariants for applications

Such invariants should be:

- Computable
- Stable
- Interpretable

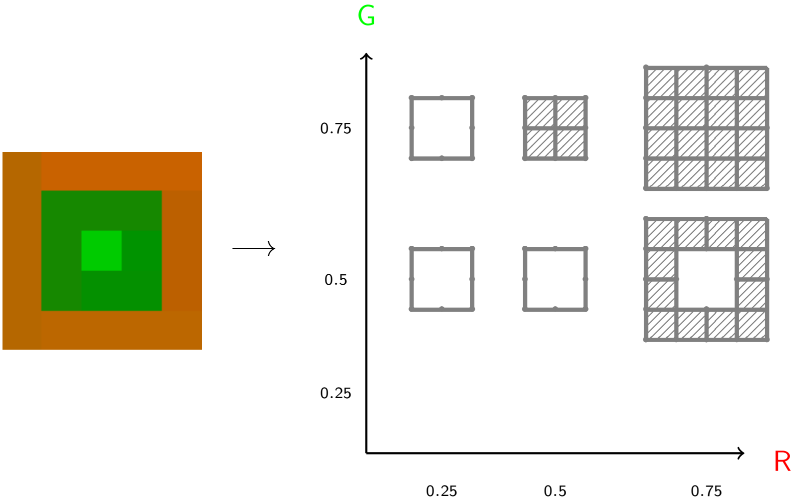
Desiderata for invariants for applications

Such invariants should be:

- Computable
- Stable
- Interpretable

From (Carlsson, Zomorodian 2009):

> *Our study of multigraded objects shows that no complete discrete invariant exists for multidimensional persistence. We still desire a discriminating invariant that captures persistent information, that is, homology classes with large persistence.*



G
0.75
0.5
0.25
R
0.25
0.5
0.75

Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

$$\mathrm{Ann}(U) = \{r \in R \mid \forall u \in U : ru = 0\}.$$

Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

$$\mathrm{Ann}(U) = \{r \in R \mid \forall u \in U : ru = 0\}.$$

A prime ideal $\mathfrak{p} \subset R$ is **associated to** M if $\mathfrak{p}$ is the annihilator of an element of M.

Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

$$\mathrm{Ann}(U) = \{r \in R \mid \forall u \in U : ru = 0\}.$$

A prime ideal $\mathfrak{p} \subset R$ is **associated to** M if $\mathfrak{p}$ is the annihilator of an element of M.

Denote by $\mathrm{Ass}(M)$ the poset of all primes associated to M.

Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

$$\mathrm{Ann}(U) = \{r \in R \mid \forall u \in U : ru = 0\}.$$

A prime ideal $\mathfrak{p} \subset R$ is **associated to** M if $\mathfrak{p}$ is the annihilator of an element of M.

Denote by $\mathrm{Ass}(M)$ the poset of all primes associated to M.

Fact: For a finitely generated $\mathbb{N}^r$-graded $\mathbb{K}[x_1, \dots, x_r]$-module M, any associated prime $\mathfrak{p}$ of M is of the form $\mathfrak{p} = (x_{i_1}, \dots, x_{i_k})$.

Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

$$\mathrm{Ann}(U) = \{r \in R \mid \forall u \in U : ru = 0\}.$$

A prime ideal $\mathfrak{p} \subset R$ is **associated to** M if $\mathfrak{p}$ is the annihilator of an element of M.

Denote by $\mathrm{Ass}(M)$ the poset of all primes associated to M.

Fact: For a finitely generated $\mathbb{N}^r$-graded $\mathbb{K}[x_1, \ldots, x_r]$-module M, any associated prime $\mathfrak{p}$ of M is of the form $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_k})$.

Definition

For such a prime $\mathfrak{p}$ define the **complement support**

$$c_{\mathfrak{p}} = \{(u_1, \ldots, u_r) \in \mathbb{N}^r \mid u_i = 0 \text{ for all } i \in \{i_1, \ldots, i_k\}\}.$$

Support shape of a module

Recall: Let M be a module over a commutative ring R. Let U be a non-empty subset of M. Define the **annihilator of** U as follows:

$$\mathrm{Ann}(U) = \{r \in R \mid \forall u \in U : ru = 0\}.$$

A prime ideal $\mathfrak{p} \subset R$ is **associated to** M if $\mathfrak{p}$ is the annihilator of an element of M.

Denote by $\mathrm{Ass}(M)$ the poset of all primes associated to M.

Fact: For a finitely generated $\mathbb{N}^r$-graded $\mathbb{K}[x_1, \dots, x_r]$-module M, any associated prime $\mathfrak{p}$ of M is of the form $\mathfrak{p} = (x_{i_1}, \dots, x_{i_k})$.

Definition

For such a prime $\mathfrak{p}$ define the **complement support**

$$c_{\mathfrak{p}} = \{(u_1, \dots, u_r) \in \mathbb{N}^r \mid u_i = 0 \text{ for all } i \in \{i_1, \dots, i_k\}\}.$$

The **support shape** of M is $ss(M) = \bigcup_{\mathfrak{p} \in Ass(M)} c_{\mathfrak{p}}$.

Stratification of the support shape

Given a sequence $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ of associated primes of M, one obtains a nested sequence

$$c_{\mathfrak{p}_m} \subset \ldots c_{\mathfrak{p}_0} \subset ss(M) \, .$$

Stratification of the support shape

Given a sequence $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ of associated primes of M, one obtains a nested sequence

$$c_{\mathfrak{p}_m} \subset \ldots c_{\mathfrak{p}_0} \subset ss(M) \, .$$

Definition

We call the poset $Ass(M)$ the **stratification** of $ss(M)$.

Stratification of the support shape

Given a sequence $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ of associated primes of M, one obtains a nested sequence

$$c_{\mathfrak{p}_m} \subset \ldots c_{\mathfrak{p}_0} \subset ss(M)\,.$$

Definition

We call the poset $Ass(M)$ the **stratification** of $ss(M)$.

Note: $ss(M)$ is completely determined by the minimal associated primes.

Example

$$M = S(-1,-1) \oplus S(-2,-2)$$

Example

$$M = S(-1,-1) \oplus S(-2,-2)$$

$$N = S(-2,-2) \oplus S(-2,-2) \oplus \frac{S(-2,-1)}{x_2} \oplus \frac{S(-1,-2)}{x_1} \oplus \frac{S(-1,-1)}{\langle x_1, x_2 \rangle}$$

Example

$$M = S(-1,-1) \oplus S(-2,-2)$$

$$N = S(-2,-2) \oplus S(-2,-2) \oplus \frac{S(-2,-1)}{x_2} \oplus \frac{S(-1,-2)}{x_1} \oplus \frac{S(-1,-1)}{\langle x_1, x_2 \rangle}$$

$Ass(M) = \{(0)\}$, and $Ass(N) = \{(0),(x_1),(x_2),(x_1,x_2)\}$.

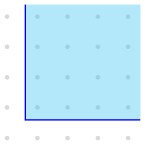
Example

$$M = S(-1,-1) \oplus S(-2,-2)$$

$$N = S(-2,-2) \oplus S(-2,-2) \oplus \frac{S(-2,-1)}{x_2} \oplus \frac{S(-1,-2)}{x_1} \oplus \frac{S(-1,-1)}{\langle x_1, x_2 \rangle}$$

$Ass(M) = \{(0)\}$, and $Ass(N) = \{(0), (x_1), (x_2), (x_1, x_2)\}$.

The points of $\mathbb{N}^r$ at which the module M, as well as N, does not vanish:



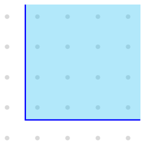
Example

$$M = S(-1,-1) \oplus S(-2,-2)$$

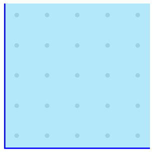
$$N = S(-2,-2) \oplus S(-2,-2) \oplus \frac{S(-2,-1)}{x_2} \oplus \frac{S(-1,-2)}{x_1} \oplus \frac{S(-1,-1)}{\langle x_1, x_2 \rangle}$$

$Ass(M) = \{(0)\}$, and $Ass(N) = \{(0), (x_1), (x_2), (x_1, x_2)\}$.

The points of $\mathbb{N}^r$ at which the module M, as well as N, does not vanish:



The support shape of M, as well as N:



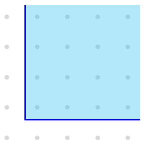
Example

$$M = S(-1,-1) \oplus S(-2,-2)$$

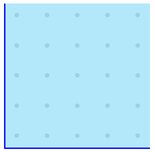
$$N = S(-2,-2) \oplus S(-2,-2) \oplus \frac{S(-2,-1)}{x_2} \oplus \frac{S(-1,-2)}{x_1} \oplus \frac{S(-1,-1)}{\langle x_1, x_2 \rangle}$$

$Ass(M) = \{(0)\}$, and $Ass(N) = \{(0), (x_1), (x_2), (x_1, x_2)\}$.

The points of $\mathbb{N}^r$ at which the module M, as well as N, does not vanish:



The support shape of M, as well as N:

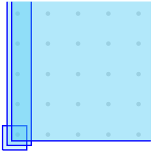


Example (cont.)

The chains in the stratification of $ss(N)$:

Example (cont.)

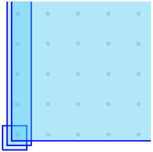
The chains in the stratification of $ss(N)$:



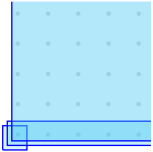
$c_{(x_1,x_2)} \subset c_{(x_1)} \subset c_{(0)}$

Example (cont.)

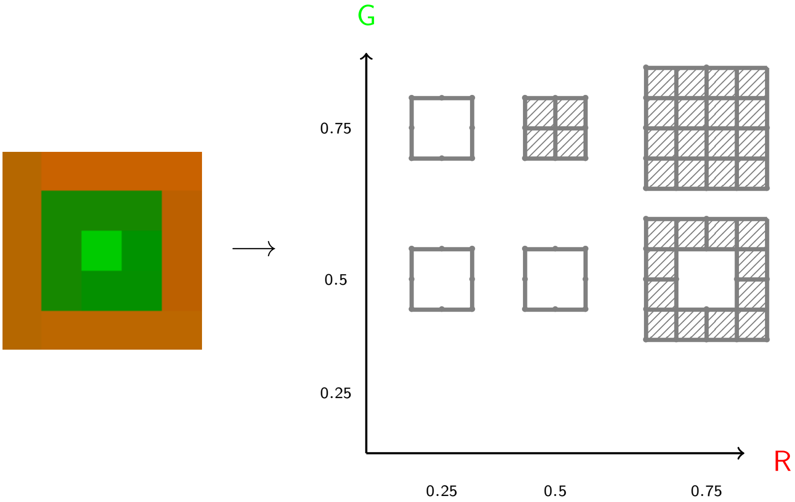
The chains in the stratification of $ss(N)$:



$c_{(x_1,x_2)} \subset c_{(x_1)} \subset c_{(0)}$



$c_{(x_1,x_2)} \subset c_{(x_2)} \subset c_{(0)}$


G
0.75
0.5
0.25
R
0.25
0.5
0.75

Generalisation of definition of birth and death

We say that a homogeneous element a of M is:

- **born at** $(u_1, \ldots, u_r) \in \mathbb{N}^r$ if the degree of a is $(u_1, \ldots, u_r)$ and a is not in the image of any sum of maps $\sum_{\mathbf{v}} x^{\mathbf{v}}$ for any $\mathbf{v} \prec \mathbf{u}$.
- If $Ann(a) \neq (0)$ let $D \subset \mathbb{N}^r$ be the subset of $\mathbb{N}^r$ obtained from the set of degrees of the set of minimal generators of $Ann(a)$ by adding to each degree the degree of a. Then we say that a **dies in degrees** D.
- If $Ann(a) = (0)$ we say that a **lives forever**.
- If $\sqrt{Ann(a)} = \langle x_{i_1}, \ldots, x_{i_k} \rangle = \mathfrak{p}$, we say that a **lives along** $c_{\mathfrak{p}} \subset \mathbb{N}^r$. In this case we say that a has **support dimension** $r - k$.

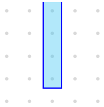
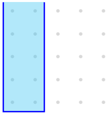
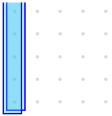
Generalisation of definition of birth and death

We say that a homogeneous element a of M is:

- **born at** $(u_1, \ldots, u_r) \in \mathbb{N}^r$ if the degree of a is $(u_1, \ldots, u_r)$ and a is not in the image of any sum of maps $\sum_{\mathbf{v}} x^{\mathbf{v}}$ for any $\mathbf{v} \prec \mathbf{u}$.
- If $Ann(a) \neq (0)$ let $D \subset \mathbb{N}^r$ be the subset of $\mathbb{N}^r$ obtained from the set of degrees of the set of minimal generators of $Ann(a)$ by adding to each degree the degree of a. Then we say that a **dies in degrees** D.
- If $Ann(a) = (0)$ we say that a **lives forever**.
- If $\sqrt{Ann(a)} = \langle x_{i_1}, \ldots, x_{i_k} \rangle = \mathfrak{p}$, we say that a **lives along** $c_{\mathfrak{p}} \subset \mathbb{N}^r$. In this case we say that a has **support dimension** $r - k$.

We call elements of support dimension 0 **transient components**, elements of support dimension $1 \leq d < r$ **persistent components**, and elements of support dimension r **fully persistent components**.

Information forgotten by $c_{\mathfrak{p}}$ and $ss(M)$

module	points at which the module does not vanish	information forgotten
$M_1 = \dfrac{S(-1,-2)}{x_1}$		"translation"
$M_2 = \dfrac{S}{x_1^2}$		"thickness"
$M_3 = \dfrac{S}{x_1} \oplus \dfrac{S}{x_1}$		"multiplicity"

Rank and local cohomology

- The rank of a finitely generated module M over an integral domain R with field of fractions F is

$$\mathrm{rk}_S M = \dim_F M \otimes_R F$$

Rank and local cohomology

- The rank of a finitely generated module M over an integral domain R with field of fractions F is

$$\mathrm{rk}_S M = \dim_F M \otimes_R F$$

  - The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.

Rank and local cohomology

- The rank of a finitely generated module M over an integral domain R with field of fractions F is

$$\mathrm{rk}_S M = \dim_F M \otimes_R F$$

  - The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.

- The 0th local cohomology of M along a prime $\mathfrak{p}$ is the set

$$H^0_{\mathfrak{p}}(M) = \{a \in M \mid \mathfrak{p}^n a = 0 \text{ for all } n \gg 0\}.$$

Rank and local cohomology

- The rank of a finitely generated module M over an integral domain R with field of fractions F is

$$\mathrm{rk}_S M = \dim_F M \otimes_R F$$

  - The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.

- The 0th local cohomology of M along a prime $\mathfrak{p}$ is the set

$$H^0_{\mathfrak{p}}(M) = \{a \in M \mid \mathfrak{p}^n a = 0 \text{ for all } n \gg 0\}.$$

  If $\mathfrak{p} = \langle x_1, \ldots, x_k \rangle$ is an associated prime of M, then
  - $H^0_{\mathfrak{p}}(M) = \{a \in M \mid a \text{ lives along } c_{\mathfrak{p}}\}$, and

Rank and local cohomology

- The rank of a finitely generated module M over an integral domain R with field of fractions F is

$$\mathrm{rk}_S M = \dim_F M \otimes_R F$$

  - The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.

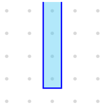
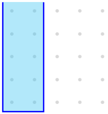
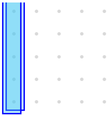
- The 0th local cohomology of M along a prime $\mathfrak{p}$ is the set

$$H^0_{\mathfrak{p}}(M) = \{a \in M \mid \mathfrak{p}^n a = 0 \text{ for all } n \gg 0\}.$$

If $\mathfrak{p} = \langle x_1, \ldots, x_k \rangle$ is an associated prime of M, then

  - $H^0_{\mathfrak{p}}(M) = \{a \in M \mid a \text{ lives along } c_{\mathfrak{p}}\}$, and
  - $H^0_{\mathfrak{p}}(M)$ is finitely generated as an $\mathbb{K}[x_{k+1}, \ldots, x_r]$-module.

Example

module	points at which the module does not vanish	
$M_1 = \frac{S(-1,-2)}{x_1}$		$\mathrm{rk}_{\mathbb{K}[x_2]} H^0_{x_1}(M_1) = 1$
$M_2 = \frac{S}{x_1^2}$		$\mathrm{rk}_{\mathbb{K}[x_2]} H^0_{x_1}(M_2) = 2$
$M_3 = \frac{S}{x_1} \oplus \frac{S}{x_1}$		$\mathrm{rk}_{\mathbb{K}[x_2]} H^0_{x_1}(M_3) = 2$

Conclusions

- The stratification of the support shape gives a summary of the existence of elements that live along certain coordinate directions.

Conclusions

- The stratification of the support shape gives a summary of the existence of elements that live along certain coordinate directions.
- The stability of the stratification probably depends on the codimension of the associated primes.

Conclusions

- The stratification of the support shape gives a summary of the existence of elements that live along certain coordinate directions.
- The stability of the stratification probably depends on the codimension of the associated primes.
- Associated primes do not give information about non-trivial second syzygies (nor higher ones).