Discussion

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Ch 5.2 Transformation of 2 RVs

X_1 & X_2 are two RV, and we have two other RVs Y_1 & Y_2 defined as follows

$$Y_1 = u_1(X_1, X_2) \qquad Y_2 = u_2(X_1, X_2)$$

And we have a joint pdf of X_1 & X_2 as $f(x_1, x_2)$

Q: How can we compute the joint pdf of Y_1 & Y_2 : $g(y_1, y_2)$

Step 0: Find the domain of Y_1 & Y_2, considering domain of X_1 & X_2 and functions u_1 & u_2

Step 1: Find the inverse functions v_1 & v_2 so that we can write

$$X_1 = v_1(Y_1, Y_2) \qquad X_2 = v_2(Y_1, Y_2)$$

Step 2: Find the Jacobian of v_1 & v_2 as

$$J = \begin{pmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{pmatrix}$$

Step 3: $g(y_1, y_2) = |J| \cdot f(v_1(y_1, y_2), v_2(y_1, y_2))$

(5.21 in PSI) Let X_1 & X_2 denote two independent RVs with $\chi^2(2)$ distribution. Define

$$Y_1 = X_1 \qquad Y_2 = X_1 + X_2$$

(a) Find the range of Y_1 & Y_2

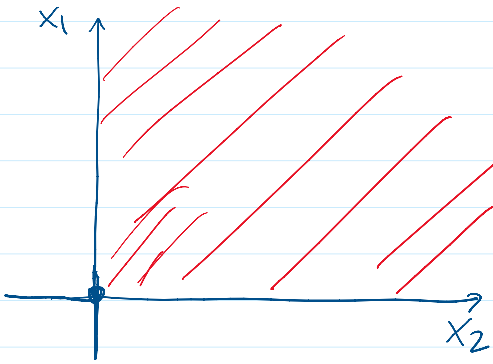
(b) Find the joint pdf of Y_1 & Y_2 : $g(y_1, y_2)$

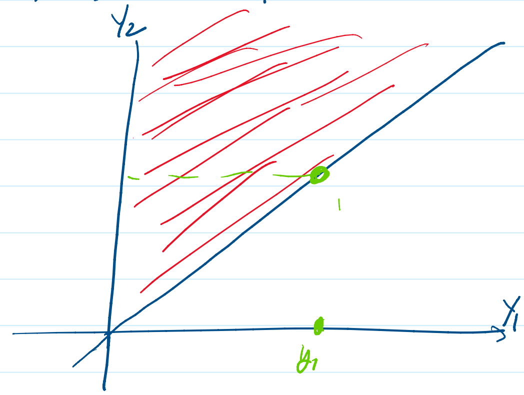
(c) Find the marginal pdf's of Y_1 & Y_2. Are Y_1 & Y_2 independent?

(a): The density of X_1 is $f_{X_1}(x_1) = \frac{1}{2} e^{-\frac{x_1}{2}}$, $f_{X_2}(x_2) = \frac{1}{2} e^{-\frac{x_2}{2}}$ $\quad 0 < x_1, x_2 < \infty$

(a): The density of X_1 is $f_{X_1}(x_1) = \frac{1}{2}e^{\frac{-x_1}{2}}$, $f_{X_2}(x_2) = \frac{1}{2}e^{\frac{-x_2}{2}}$ $0 < x_1, x_2 < \infty$

$Y_1 = X_1 \in (0, \infty)$ $Y_2 = X_1 + X_2 \in (0, \infty)$ $Y_2 = X_1 + X_2 = Y_1 + X_2 > Y_1$





(b) $f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$ because X_1 & X_2 are independent.

$$= \frac{1}{2}e^{-\frac{x_1}{2}} \cdot \frac{1}{2}e^{-\frac{x_2}{2}} = \frac{1}{4}e^{-\frac{x_1+x_2}{2}} \qquad 0 < x_1, x_2 < \infty.$$

Step 1) $Y_1 = X_1 = u_1(X_1, X_2)$ $Y_2 = X_1 + X_2 = u_2(X_1, X_2)$

$X_1 = Y_1 = v_1(Y_1, Y_2)$ $X_2 = \underbrace{(X_1 + X_2)}_{Y_2} - \underbrace{X_1}_{Y_1} = Y_2 - Y_1$

$v_1(y_1, y_2) = y_1$

$v_2(y_1, y_2) = y_2 - y_1$

Step 2: $$J = \begin{pmatrix} \frac{\partial v_1}{\partial y_1} & \frac{\partial v_1}{\partial y_2} \\ \frac{\partial v_2}{\partial y_1} & \frac{\partial v_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$|J| = 1 \cdot 1 - 0 \cdot (-1) = 1.$

↳ determinant of matrix

Step 3 $g(y_1, y_2) = |J| \, f(v_1(y_1, y_2), v_2(y_1, y_2))$

$$= 1 \, f(y_1, y_2 - y_1) \quad \left(f(x_1, x_2) = \frac{1}{4}e^{\frac{-x_1+x_2}{2}} \right.$$

$$= 1\, f(y_1, y_2 - y_1) \quad \left(f(x_1, x_2) = \frac{1}{4} e^{-\frac{x_1+x}{2}} \right.$$

$$= \frac{1}{4} e^{-\frac{y_1 + (y_2 - y_1)}{2}} = \frac{1}{4} \cdot e^{\frac{-y_2}{2}}$$

$$g(y_1, y_2) = \frac{1}{4} e^{\frac{-y_2}{2}} \quad \text{for} \quad 0 < y_1 < y_2 < \infty$$

(c) $$g_{Y_1}(y_1) = \int_{y_1}^{\infty} g(y_1, y_2)\, dy_2 = \int_{y_1}^{\infty} \frac{1}{4} e^{\frac{-y_2}{2}}\, dy_2 = \frac{-e^{\frac{-y_2}{2}}}{2} \Bigg|_{y=y_1}^{\infty} =$$

$$= \underbrace{\frac{-e^{-\frac{\infty}{2}}}{2}}_{0} - \left(- \frac{e^{\frac{-y_1}{2}}}{2} \right) = \frac{1}{2} e^{\frac{-y_1}{2}}$$

$$g_{Y_1}(y_1) = \frac{1}{2} e^{\frac{-y_1}{2}}$$

$$g_{Y_2}(y_2) = \int_0^{y_2} g(y_1, y_2)\, dy_1 = \int_0^{y_2} \frac{1}{2} e^{\frac{-y_2}{2}}\, dy_1 = \frac{1}{2} y_2 e^{\frac{-y_2}{2}}$$

$$g_{Y_2}(y_2) = \frac{1}{2} y_2 e^{\frac{-y_2}{2}} \qquad 0 < y_2 < \infty$$

Q: Are Y_1 & Y_2 independent? NO. Because if Y_1 & Y_2 would be independent, then we would have $g_{Y_1}(y_1) \cdot g_{Y_2}(y_2) = g(y_1, y_2)$

$$\frac{1}{2} e^{\frac{-y_1}{2}} \; \frac{1}{2} y_2 e^{\frac{-y_2}{2}} \neq \frac{1}{4} e^{\frac{-y_2}{2}}$$

Not independent!

Ex2: Assume that the high school grade point average of a student, X, and college freshmen gpa of a student, Y, distributed according to a bivariate normal distribution with parameters

$$\mu_X = 3, \quad \mu_Y = 4.3, \quad \sigma_X = 0.4, \quad \sigma_Y = 1.25, \quad EXY = 13.3$$

(a) Find $\rho = \rho(X,Y)$

Sol $\rho = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

$$Cov(X,Y) = EXY - EX\,EY$$
$$= 13.3 - 3 \cdot 4.3$$
$$= 13.3 - 12.9$$
$$= 0.4$$

$$= \frac{\cancel{0.4}}{\cancel{0.4} \times 1.25} = \frac{1}{\frac{5}{4}} = 0.8$$