Ch 5.2 Transformation of 2 RVs

$X_1$ and $X_2$ are two RVs, and we have two other RVs $Y_1$ and $Y_2$ defined as follows:

$$Y_1 = U_1(X_1, X_2) \hspace{1cm} Y_2 = U_2(X_1, X_2)$$

And we have a joint pdf of $X_1$ and $X_2$ as $f(x_1, x_2)$.

Q: How can we compute the joint pdf of $Y_1$ and $Y_2$: $g(y_1, y_2)$?

Step 0: Find the domain of $Y_1$ and $Y_2$, considering domain of $X_1$ and $X_2$ and functions $U_1$ and $U_2$.

Step 1: Find the inverse functions $U_1^{-1}$ and $U_2^{-1}$ so that we can write

$$X_1 = U_1^{-1}(Y_1, Y_2) \hspace{1cm} X_2 = U_2^{-1}(Y_1, Y_2)$$

Step 2: Find the Jacobian of $U_1$ and $U_2$ as

$$J = \begin{pmatrix}
\frac{\partial U_1}{\partial Y_1} & \frac{\partial U_1}{\partial Y_2} \\
\frac{\partial U_2}{\partial Y_1} & \frac{\partial U_2}{\partial Y_2}
\end{pmatrix}$$

Step 3: $g(y_1, y_2) = \left| J \right| \cdot f(U_1^{-1}(y_1, y_2), U_2^{-1}(y_1, y_2))$

(S.21 in PSI) Let $X_1$ and $X_2$ denote two independent RVs with $X^2(2)$ distribution. Define

$$Y_1 = X_1 \hspace{1cm} Y_2 = X_1 + X_2$$

(a) Find the range of $Y_1$ and $Y_2$.

(b) Find the joint pdf of $Y_1$ and $Y_2$: $g(y_1, y_2)$.

(c) Find the marginal pdf's of $Y_1$ and $Y_2$. Are $Y_1$ and $Y_2$ independent?

(a): The density of $X_1$ is $f_{X_1}(x_1) = \frac{1}{2}e^{-\frac{x_1^2}{2}}, \hspace{1cm} f_{X_2}(x_2) = \frac{1}{2}e^{-\frac{x_2^2}{2}} \hspace{1cm} 0 < x_1, x_2 < \infty$
(a) The density of $X_i$ is $f_{X_i}(x_i) = \frac{1}{2} e^{-\frac{x_i}{2}}, \quad f_{X_2}(x_2) = \frac{1}{2} e^{-\frac{x_2}{2}} \quad 0 < x_1, x_2 < \infty$

$Y_1 = X_1 \in (0, \infty) \quad Y_2 = X_1 + X_2 \in \mathbb{R}_+$ \quad Y_2 = X_1 + X_2 = Y_1 + X_2 > Y_1$

(b) $f(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$ because $X_1$ & $X_2$ are independent.

$$= \frac{1}{2} e^{-\frac{x_1}{2}} \cdot \frac{1}{2} e^{-\frac{x_2}{2}} = \frac{1}{4} e^{-\frac{x_1 + x_2}{2}} \quad 0 < x_1, x_2 < \infty.$$

Step 1: $Y_1 = X_1 = u_1(y_1, y_2) \quad Y_2 = X_1 + X_2 = u_2(y_1, y_2)$

$X_1 = Y_1 = u_1(y_1, y_2) \quad X_2 = \frac{(X_1 + X_2) - X_1}{Y_2} = \frac{Y_2}{Y_1}$

$u_1(y_1, y_2) = y_1 \quad u_2(y_1, y_2) = y_2 - y_1$

Step 2: $J = \begin{pmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$|J| = 1 \cdot 1 - 0 \cdot (-1) = 1.$

Determinant of matrix

Step 3: $g(y_1, y_2) = |J| f(u_1(y_1, y_2), u_2(y_1, y_2))$

$$= 1 \cdot f(y_1, y_2 - y_1) \cdot \frac{1}{4} e^{-\frac{x_1 + x_2}{2}}$$
\[
\begin{align*}
&= 1 \cdot \frac{1}{4} e^{-\frac{x_1^2 + x_2^2}{2}} \\
&= \frac{1}{2} e^{-\frac{y_1 + (y_2 - y_1)}{2}} = \frac{1}{2} e^{-\frac{y_2}{2}}
\end{align*}
\]

\[ g(y_1, y_2) = \frac{1}{2} e^{-\frac{y_2}{2}} \quad \text{for} \quad 0 < y_1 < y_2 < \infty \]

\[(e) \quad g_{y_1}(y_1) = \int_{y_1}^{\infty} g(y_1, y_2) \, dy_2 = \int_{y_1}^{\infty} \frac{1}{2} e^{-\frac{y_2}{2}} \, dy_2 = -\frac{e^{-\frac{y_1^2}{2}}}{2} \bigg|_{y_2 = y_1}^{\infty} = \]

\[= -\frac{1}{2} e^{-\frac{y_1^2}{2}} - \left( -\frac{e^{-\frac{y_1}{2}}}{2} \right) = \frac{1}{2} e^{-\frac{y_1}{2}} \]

\[g_{y_2}(y_2) = \frac{1}{2} e^{-\frac{y_2}{2}} \quad 0 < y_2 < \infty \]

Q: Are \( Y_1 \) and \( Y_2 \) independent? NO. Because if \( Y_1 \) and \( Y_2 \) would be independent, then we would have \( g_{y_1}(y_1) \cdot g_{y_2}(y_2) = g(y_1, y_2) \), but

\[\frac{1}{2} e^{-\frac{y_1^2}{2}} \cdot \frac{1}{2} y_2 e^{-\frac{y_2}{2}} = \frac{1}{4} e^{-\frac{y_2}{2}} \quad \text{Not independent!} \]
EX2: Assume that the high school grade point average of a student, $X$, and college freshman gpa of a student, $Y$, distributed according to a bivariate normal distribution with parameters

$$M_X = 3, \quad M_Y = 4.3, \quad \sigma_X = 0.4, \quad \sigma_Y = 1.25, \quad E(XY) = 13.3$$

(b) Find $\rho = \rho(X, Y)$

**Sol**

$$\rho = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y)$$

$$= 13.3 - 3 \cdot 4.3$$

$$= 13.3 - 12.9$$

$$= 0.4$$

$$\frac{0.4}{0.4 \times 1.25} = \frac{1}{\frac{5}{4}} = 0.8$$