Discussion

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(4.13 – modified) Let the joint pmf of RVs X & Y be

$$f(x,y) = c(x+y) \quad x = 1,2; \; y = 1,2,3$$

(a) Find c A) $\frac{1}{18}$ B) $\frac{1}{19}$ C) $\frac{1}{21}$ D) $\frac{1}{24}$ E) $\frac{1}{30}$

(b) Compute $f_X(x)$ (Marginal pmf of X)

A) $\frac{2x+y}{3}$ B) $\frac{2x+1}{7}$ C) $\frac{x+1}{7}$ D) $\frac{x+2}{7}$ E) $\frac{x+3}{9}$

Sol (a) $\sum_{x=1}^{2}\sum_{y=1}^{3} f(x,y) = 1$

$$1 = \sum_{x=1}^{2}\underbrace{\sum_{y=1}^{3} c(x+y)} = c\sum_{x=1}^{2}\left((x+1)+(x+2)+(x+3)\right)$$

$$= c\sum_{x=1}^{2}(3x+6)$$

$$= c(9+12)$$

$$= 21c$$

$$1 = 21c \Rightarrow c = \frac{1}{21}$$

Second way:

$f(1,1) = 2c$

$f(1,2) = f(2,1) = 3c$

$f(2,2) = f(1,3) = 4c$

$f(2,3) = 5c$

Add all 6 values up $\Rightarrow 21c = 1$

(b) $f_X(x) = \sum_{y=1}^{3} f(x,y) = \sum_{y=1}^{3} \frac{1}{21}(x+y)$

(b) $f_X(x) = \sum_{y=1} f(x,y) = \sum_{y=1} \frac{1}{21}(x+y)$

$$= \frac{1}{21}(x+1) + \frac{1}{21}(x+2) + \frac{1}{21}(x+3)$$

$$= \frac{1}{21}(3x+6) = \frac{x+2}{7}$$

(c) $f_Y(y) = ?$

A) $\frac{2y+3}{21}$ B) $\frac{y+6}{21}$ C) $\frac{y+1}{7}$ D) $\frac{2y+1}{14}$ E) $\frac{3y+1}{21}$

(d) Compute $P(X>Y)$ A) $\frac{2}{21}$ B) $\frac{1}{7}$ C) $\frac{4}{21}$ D) $\frac{5}{21}$ E) $\frac{2}{7}$

Sol (c) $f_Y(y) = \sum_{x=1}^{2} f(x,y) = \sum_{x=1}^{2} \frac{1}{21}(x+y) = \frac{1}{21}\left((1+y)+(2+y)\right)$

$$= \frac{2y+3}{21}$$

Sol (d) $P(X>Y) = P(X=2, Y=1) = f(2,1) = \frac{1}{21}(2+1) = \frac{1}{7}$

(e) Compute $Cov(X,Y)$. Are X & Y independent?

A) $\frac{1}{49}$ B) $\frac{3}{49}$ C) $\frac{-3}{49}$ D) $\frac{5}{147}$ E) $\frac{-2}{147}$

Sol (e) Recall $Cov(X,Y) = EXY - EX\,EY$

$$EX = \mu_X = \sum_{x=1}^{2} x f_X(x) = \sum_{x=1}^{2} x \cdot \frac{x+2}{7} = 1 \cdot \frac{3}{7} + 2 \cdot \frac{4}{7} = \frac{11}{7}$$

$\sum^{3} \quad \sum^{3} \quad \frac{2y+3}{} \quad 5 \quad 7 \quad 9 \quad = \frac{46}{}$

$$EY = \mu_Y = \sum_{y=1}^{3} y\, f_Y(y) = \sum_{y=1}^{3} y \cdot \frac{2y+3}{21} = 1\cdot\frac{5}{21} + 2\cdot\frac{7}{21} + 3\cdot\frac{9}{21} = \frac{46}{21}$$

$$EXY = \sum_{x=1}^{2}\sum_{y=1}^{3} xy\, f(x,y)$$

$$= \sum_{x=1}^{2}\sum_{y=1}^{3} xy\, \frac{1}{21}(x+y)$$

$$= \frac{1}{21}\left(1\cdot 1\cdot(2) + 1\cdot 2\cdot(3) + 1\cdot 3\cdot(4) + 2\cdot 1\cdot 3 + 2\cdot 2\cdot 4 + 2\cdot 3\cdot 5\right)$$

$$= \frac{1}{21}(72) = \frac{24}{7}$$

$$Cov(X,Y) = EXY - EXEY$$

$$= \frac{24}{7} - \frac{11}{7}\cdot\frac{46}{21}$$

$$= \frac{24\cdot 21 - 11\cdot 46}{147} = \frac{-2}{147}$$

X & Y are not independent because $Cov(X,Y) \neq 0$

(f) Compute $g(x|y)$ = conditional distribution of X, given Y=y.

Sol: $$g(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{1}{21}(x+y)}{\frac{2y+3}{21}} = \frac{x+y}{2y+3}$$

$\frac{\quad}{21}$

$$g(1 \mid y) = \frac{1+y}{2y+3}$$

$$g(2 \mid y) = \frac{2+y}{2y+3}$$

(Ex2) let $f(x,y) = c\,x^2 y^3 \quad -1 \le x \le 1, \quad 0 \le y \le 1$ be the joint pdf of continuous RVs X & Y.

(a) Find c A) 6 B) 8 C) 12 D) 4 E) 1

(b) $f_X(x) = ?$ A) $\frac{x^2}{2}$ B) x^2 C) $\frac{2x^3}{3}$ D) $\frac{x^3}{4}$ E) None.

Sol. $1 = \int_0^1 \int_{-1}^1 f(x,y)\,dx\,dy$

$$= c \int_0^1 \int_{-1}^1 \underbrace{x^2 y^3}\,dx\,dy$$

$$= c \int_0^1 \left(\frac{x^3}{3} y^3 \Big|_{x=-1}^{1} \right) dy$$

$$= c \int_0^1 \underbrace{\left(\frac{1}{3} y^3 - \left(-\frac{1}{3} y^3 \right) \right)}_{\frac{2}{3} y^3} dy = \frac{2}{3} c \underbrace{\int_0^1 y^3\,dy}. =$$

$$= \frac{2}{3} c \cdot \frac{y^4}{4} \Big|_{y=0}^{1} = \frac{2}{3} c \cdot \frac{1}{4} = 1 \Rightarrow \underline{\underline{c=6}}$$

(b) $f_X(x) = \int_0^1 f(x,y)\, dy$

$$= \int_0^1 6 x^2 y^3\, dy = 6x^2 \underbrace{\left(\frac{y^4}{4} \Big|_{y=0}^{1} \right)}_{\frac{1}{4}} = \frac{3x^2}{2}$$

$f_Y(y) = 4y^3 \quad \Rightarrow \boxed{f(x,y) = f_X(x)\, f_Y(y)}$

$\Rightarrow$ X & Y are independent.