Discussion

Thursday, April 29, 2021 1:25 PM

Ex Two independent dice are thrown. Let random variable (RV) X denote the sum of the numbers facing up. Find.

(a) Pmf of X (draw bar graph)

(b) Compute $E(X)$ A) 6 B) 7 C) 8 D) 9 E) 14.

(c) $M(t) = Mgf_X(t)$

Sol Denote the outcomes of two dices as D_1 & D_2 $\Rightarrow X = D_1 + D_2$

What is sample space of $X = S_X = \{2, 3, \ldots, 12\}$

$\forall x \in S_X, \quad f_X(x) := P(X = x)$

$$f_X(2) = P(X=2) = P((D_1, D_2) = (1,1)) = \frac{1}{36}$$

$$f_X(3) = P(X=3) = P((D_1, D_2) \in \{(1,2) \text{ or } (2,1)\}) = \frac{2}{36}$$

$$f_X(4) = P(X=4) = P((D_1, D_2) \in \{(1,3), (2,2), (3,1)\}) = \frac{3}{36}$$

$\vdots$

$$f_X(7) = P(X=7) = P((D_1, D_2) \in \{(1,6), (2,5), \ldots, (6,1)\} = \frac{6}{36}$$

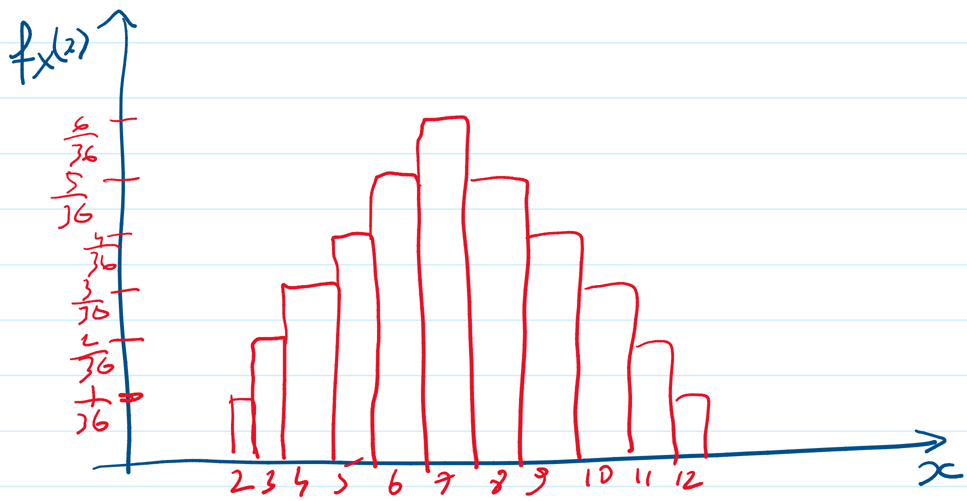
$$f_X(8) = P(X=8) = P((D_1, D_2) \in \{(2,6), (3,5), (4,4), (5,3), (6,2)\} = \frac{5}{36}$$

$$f_X(9) = \ldots = \frac{4}{36}$$

$\vdots$

$\vdots$

$$f_X(12) = P(X=12) = P(D_1=6, D_2=6) = \frac{1}{36}$$



(b) $$EX = \sum_{x \in S_X} x\, f_X(x)$$

$$= \sum_{x=2}^{12} x \cdot f_X(x)$$

$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + \cdots + 12 \cdot \frac{1}{36}$$

$$= 7.$$

Usually, we say $EX = E(D_1 + D_2) = ED_1 + ED_2 = \frac{7}{2} + \frac{7}{2} = 7$

$$ED_1 = \sum_{x \in S_{D_1}} x \cdot f_{D_1}(x) = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$

(c) $M(t) = E(e^{tX})$ $\quad$ t is variable, X is RV, e is euler number.

In general, $E(g(X)) = \sum_{x \in S_X} g(x) \cdot f_X(x)$

x is variable, X is R.V.

e^{tX} is also a function of X. $\quad e^{tX} = g(X)$

$$M(t) = E(e^{tX}) = \sum_{x \in S_X} e^{tx} \cdot \underbrace{f_X(x)}_{P(X=x)}$$

In our experiment, $S_X = \{2, 3, \ldots, 12\}$

$$= \sum_{x=2}^{12} e^{tx} \cdot P(X=x)$$

$$= e^{2t}P(X=2) + e^{3t}P(X=3) + e^{4t}P(X=4) + \cdots + e^{12t}P(X=12)$$

$$= \underbrace{\frac{1}{36}}_{P(X=2)} \cdot e^{2t} + \underbrace{\frac{2}{36}}_{P(X=3)} e^{3t} + \underbrace{\frac{3}{36}}_{P(X=4)} e^{4t} + \cdots + \frac{6}{36} \cdot e^{7t} + \frac{5}{36} e^{8t} + \cdots + \underbrace{\frac{1}{36}}_{P(X=12)} e^{12t}.$$

In a way, pmf is encoded in the mgf.

Q: Why we care about mgf.?

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A: It is useful to compute mean and variance.

$$M(t) = E(e^{tX})$$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \frac{t^4X^4}{4!} + \cdots\right) \quad \left(\begin{array}{l}\text{Taylor expansion}\\ \text{of } e^{tX}\end{array}\right)$$

$$= E(1) + E(tX) + E\left(\frac{t^2}{2}X^2\right) + E\left(\frac{t^3}{3!}X^3\right) + \cdots$$

(Recall $E(aX+b) = aEX + b$)

$$M(t) = 1 + tEX + \frac{t^2}{2!}EX^2 + \frac{t^3}{3!}EX^3 + \cdots$$

$$M(0) = 1 + 0 = 1$$

$$M'(t) = EX + \frac{t}{1!}EX^2 + \frac{t^2}{2!}EX^3 + \frac{t^3}{3!}EX^4 + \cdots$$

$$M'(0) = EX$$

$$M''(t) = EX^2 + \frac{t}{1!}EX^3 + \frac{t^2}{2!}EX^4 + \frac{t^3}{3!}EX^5 + \cdots$$

$$M''(0) = EX^2$$

$$Var(X) = EX^2 - (EX)^2 = M''(0) - (M'(0))^2$$

Ex 2 (old quiz problem) X is a discrete RV with mgf

$$M(t) = \frac{3e^t}{4}\left(1 - \frac{e^t}{4}\right)^{-1} = \frac{\frac{3}{4}e^t}{1 - \frac{e^t}{4}}$$

$$M(t) = \frac{3e^t}{4}\left(1-\frac{e^t}{4}\right) = \frac{\quad}{1-\frac{e^t}{4}}$$

A) Compute EX A) 1 B) $\frac{4}{3}$ C) $\frac{3}{4}$ D) $\frac{8}{3}$ E) $\frac{1}{2}$

B) Compute $Var(X)$ A) 1 B) $\frac{2}{3}$ C) $\frac{3}{2}$ D) $\underline{\underline{\frac{4}{3}}}$ E) None.

Sol

(a) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ $\qquad M(t) = \frac{\overset{\to f}{\frac{3}{4}e^t}}{\underset{\to g}{1-\frac{e^t}{4}}}$

$$M'(t) = \frac{\frac{3}{4}e^t\left(1-\frac{e^t}{4}\right) - \frac{3}{4}e^t\left(-\frac{1}{4}e^t\right)}{\left(1-\frac{1}{4}e^t\right)^2} = \frac{\frac{3}{4}e^t}{\left(1-\frac{1}{4}e^t\right)^2}$$

$$EX = M'(0) = \frac{\frac{3}{4}\cdot 1}{\left(1-\frac{1}{4}\cdot 1\right)^2} = \frac{\frac{3}{4}}{\left(\frac{3}{4}\right)^2} = \frac{4}{3}$$

(b) $M''(t) = \left(\frac{\overset{\to f}{\frac{3}{4}e^t}}{\underset{\to g}{\left(1-\frac{1}{4}e^t\right)^2}}\right)' =$

$$= \frac{\frac{3}{4}e^t\left(1-\frac{1}{4}e^t\right)^2 - \frac{3}{4}e^t\, 2\left(1-\frac{1}{4}e^t\right)'\left(-\frac{1}{4}e^t\right)}{\left(1-\frac{1}{4}e^t\right)^4}$$

$$M''(0) = \frac{\frac{3}{4}\cdot\left(1-\frac{1}{4}\right)^2 - \frac{3}{4}\cdot 1\cdot 2\left(1-\frac{1}{4}\right)\left(-\frac{1}{4}\right)}{\left(1-\frac{1}{4}\right)^4}$$

$$= \frac{\frac{3}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{3}{4}\cdot 2\cdot\frac{3}{4}\cdot\frac{1}{4}}{\quad}$$

$$= \frac{\frac{3}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{3}{4}\cdot 2 - \frac{3}{4}\cdot\frac{1}{4}}{\left(\frac{3}{4}\right)^4}$$

$$= \frac{\frac{3}{4} + 2 - \frac{1}{4}}{\left(\frac{3}{4}\right)^2} = \frac{\frac{5}{4}}{\frac{9}{16}} = \frac{20}{9}$$

$M''(0) = EX^2 = \frac{20}{9}$ $\quad$ $Var(X) = EX^2 - (EX)^2 = \frac{20}{9} - \left(\frac{4}{3}\right)^2 = \boxed{\frac{4}{9}}$

PM-P2 EX3

Suppose that a coin is not fair so that the probability of obtaining a head is $p \in (0,1)$.

(a) [5pts.] On average, how many flips are needed to obtain a head?

X = # of flips we need to make the first heads.

$P(X=1) = P(H) = p$

$P(X=2) = P(TH) = (1-p)p$

$P(X=3) = P(TTH) = (1-p)^2 \cdot p$

$\vdots$

$P(X=n) = P(\underbrace{TT\cdots T}_{n-1}H) = (1-p)^{n-1}\cdot p$ $\quad \forall n = 1,2,3,\ldots$

$\vdots$

X is geometric RV with parameter p.

$$M(t) = E(e^{tX}) = \sum_{n\in S_X} e^{tn} P(X=n)$$

$$= e^{t} \cdot P(X=1) + e^{2t} P(X=2) + e^{3t} P(X=3) + \cdots$$

$$= e^{t} p + e^{2t} \cdot (1-p) p + e^{3t} (1-p)^2 \cdot p + e^{4t} (1-p)^3 p + \cdots$$

$$= p e^{t} \left(1 + e^{t}(1-p) + e^{2t}(1-p)^2 + e^{3t}(1-p)^3 + \cdots \right)$$

$$1 + r + r^2 + r^3 + \cdots$$

$$r = |e^{t}(1-p)| < 1 \qquad e^{t} < \frac{1}{1-p} \qquad t < \ln\left(\frac{1}{1-p}\right)$$

$$= p e^{t} \cdot \frac{1}{1 - e^{t}(1-p)} = \frac{p e^{t}}{1 - e^{t}(1-p)}$$

$M(t) = \dfrac{p e^{t}}{1 - e^{t}(1-p)}$ is Mgf of X.

$$EX = M'(0)$$

$$M'(t) = \frac{p e^{t} (1 - e^{t}(1-p)) - p e^{t} (-e^{t}(1-p))}{(1 - e^{t}(1-p))^2}$$

$$= \frac{p \cdot e^{t}}{(1 - e^{t}(1-p))^2}$$

$M'(0) = \dfrac{p}{(1-(1-p))^2} = \dfrac{p}{p^2} = \dfrac{1}{p} = EX =$ mean of geometric RV.