Discussion

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Example 1.16. (PSI 1.5.9 - modified) There is a new diagnostic test for *coronavirus* that occurs in about 10% of the population. The test is not perfect, but will detect a person with the disease 95% of the time. It will, however, say that a person without the disease has the disease about 20% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What is the conditional probability that the person actually has the disease?

(A) $\frac{1}{3}$ (B) $\frac{9}{100}$ (C) $\frac{19}{55}$ (D) $\frac{15}{66}$ (E) $\frac{33}{95}$

Sol. Say S is the selected person uniformly random among population.

$D = \{S \text{ has the disease}\}$

$T = \{\text{The test is positive}\}$

$\Rightarrow P(D \mid T)$ → information provided

$P(D) = 0.1$ $\quad P(T \mid D) = 0.95$

$P(T \mid D^c) = 0.2$

$$P(D \mid T) = \frac{P(D \cap T)}{P(T)}$$

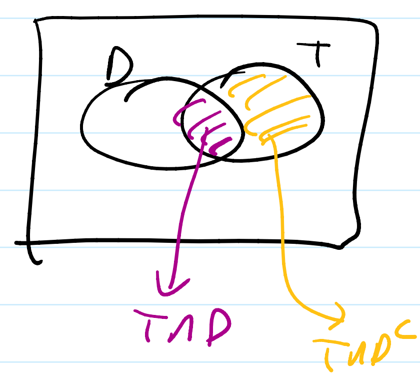
$$= \frac{P(D\cap T)}{P(D \cap T) + P(D^c \cap T)}$$

$$= \frac{P(T \mid D) \cdot P(D)}{P(T\mid D)\cdot P(D) + P(T \mid D^c)\cdot P(D^c)}$$

$$= \frac{0.95 \cdot 0.1}{0.95 \cdot 0.1 + 0.2 \cdot 0.9}$$

$P(D^c) = 1 - P(D) = 0.9$

$$= \frac{0.095}{0.095 + 0.18} = \frac{95}{275} = \frac{19}{55}$$



$P(A \mid B) = \frac{P(A\cap B)}{P(B)}$

$\Rightarrow P(A \cap B) = P(A \mid B) \cdot P(B)$

$$= \frac{0.095}{0.035+0.18} = \frac{95}{275} = \boxed{\frac{19}{55}}$$

Note $$P(D_1 \mid T) = \frac{P(T \mid D_1)\cdot P(D_1)}{P(T)}$$

$$= \frac{P(T \mid D_1)\cdot P(D_1)}{P(T\cap D_1)+P(T\cap D_2)+\cdots+P(T\cap D_n)}$$

$$= \frac{P(T \mid D_1)\cdot P(D_1)}{P(T\mid D_1)\cdot P(D_1)+P(T\mid D_2)\cdot P(D_2)+\cdots+P(T\mid D_n)\cdot P(D_n)}$$

$D_1 \cup D_2 \cup \cdots D_{n-1} \cup D_n = \Omega$, disjoint

$n=2$ $\quad D_1 \cup D_2 = \Omega$

$\quad D \cup D^c = \Omega$

Ex2 (2.13) For each of the following, determine the constant c so that $f(x)$ satisfies the condition of being a pmf of some RV X.

(a) $f(x) = \frac{x}{c}$ $\quad x = 1,2,3,4$ $\quad$ A) 1 B) 3 C) 4 D) 6 E) 10

$x \in S_x = \{1,2,3,4\}$

(b) $f(x) = c\cdot\frac{1}{4^x}$ $\quad x = 1,2,3,\ldots$ $\quad$ A) 3 B) $\frac{1}{3}$ C) 4 D) $\frac{1}{4}$ E) 9

$x \in S_x = \{1,2,3,4,\ldots\}$

Sol

2 conditions: $f(x) \geq 0$, $\quad \sum_{x\in S_x} f(x) = 1$

$4 \qquad\qquad 4$

(a) $1 = \sum_{x \in S_X} f(x) = \sum_{x=1}^{4} f(x) = \sum_{x=1}^{4} \frac{x}{c} = \frac{1}{c} + \frac{2}{c} + \frac{3}{c} + \frac{4}{c} = \frac{10}{c}$

$\underline{\underline{c=10}}$

(b) $1 = \sum_{x \in S_X} f(x) = \sum_{x=1}^{\infty} \frac{c}{4^x} = c \sum_{x=1}^{\infty} \frac{1}{4^x}$

$$= c\Big(\underbrace{\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots}_{S = \frac{1}{3}} \Big)$$

$$S = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots$$

$$\frac{S}{4} = \qquad \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + \cdots$$

$$-$$

$$\frac{3S}{4} = \frac{1}{4} \qquad 3S = 1 \qquad S = \frac{1}{3}$$

$$1 = cS = c \cdot \frac{1}{3} \implies c = 3$$

Geometric series formula: $a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ $(|r| < 1)$

$$1 = \sum_{x=1}^{\infty} \frac{c}{4^x} = \frac{c}{4} + \frac{c}{4^2} + \frac{c}{4^3} + \cdots = \frac{\frac{c}{4}}{1 - \frac{1}{4}} = \frac{c}{3}$$

$$a = \frac{c}{4} \qquad r = \frac{1}{4}$$

Ex Two independent dice are thrown. Let random variable (RV) X denote the sum of the numbers facing up. Find.

(a) Pmf of X (draw bar graph)

(b) Compute $E(X)$ A) 6 B) 7 C) 8 D) 9 E) 14.

(c) $M(t) = Mgf_X(t)$

Sol

Denote the outcomes of two dices as D_1 & D_2 $\Rightarrow X = D_1 + D_2$

What is sample space of $X = S_X = \{2, 3, \ldots, 12\}$

$\forall x \in S_X, \quad f_X(x) := P(X = x)$

$f_X(2) = P(X=2) = P((D_1, D_2) = (1,1)) = \frac{1}{36}$

$f_X(3) = P(X=3) = P((D_1, D_2) \in \{(1,2) \text{ or } (2,1)\}) = \frac{2}{36}$

$f_X(4) = P(X=4) = P((D_1, D_2) \in \{(1,3), (2,2), (3,1)\}) = \frac{3}{36}$

$\vdots$

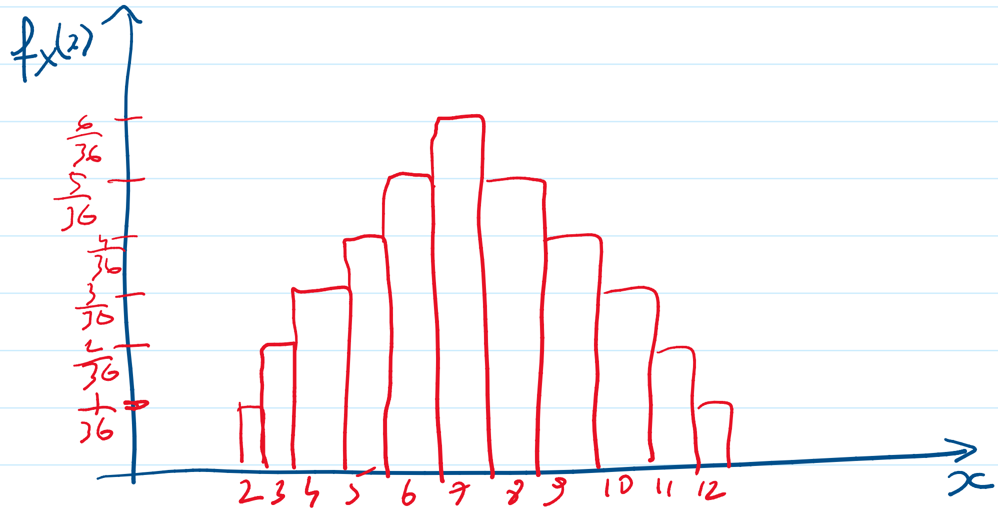
$f_X(7) = P(X=7) = P((D_1, D_2) \in \{(1,6), (2,5), \ldots, (6,1)\}) = \frac{6}{36}$

$f_X(8) = P(X=8) = P((D_1, D_2) \in \{(2,6), (3,5), (4,4), (5,3), (6,2)\}) = \frac{5}{36}$

$f_X(9) = \text{— — — — — —} = \frac{4}{36}$

$\vdots$

$$f_X(12) = P(X=12) = P(D_1=6, D_2=6) = \frac{1}{36}$$



(b) $$EX = \sum_{x \in S_X} x\, f_X(x)$$

$$= \sum_{x=2}^{12} x \cdot f_X(x)$$

$$= 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + \cdots + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} + \cdots + 12 \cdot \frac{1}{36}$$

$$= 7.$$

Usually, we say $EX = E(D_1 + D_2) = ED_1 + ED_2 = \frac{7}{2} + \frac{7}{2} = 7$

$$ED_1 = \sum_{x \in S_{D_1}} x \cdot f_{D_1}(x) = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2}$$