Discussion
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Example 1.16. (PSI 1.5.9 - modified) There is a new diagnostic test for coronavirus that occurs in about 10% of the population. The test is not perfect, but will detect a person with the disease 95% of the time. It will, however, say that a person without the disease has the disease about 20% of the time. A person is selected at random from the population, and the test indicates that this person has the disease. What is the conditional probability that the person actually has the disease?

\[
\begin{align*}
&\text{(A) } \frac{1}{3} \quad \text{(B) } \frac{9}{100} \quad \text{(C) } \frac{19}{55} \quad \text{(D) } \frac{15}{66} \quad \text{(E) } \frac{33}{55}
\end{align*}
\]

Solution

So, let’s select a person uniformly random from the population.

\[
\begin{align*}
D &= \{ \text{S has the disease} \} \quad \text{Information provided} \\
T &= \{ \text{The test is positive} \} \\
P(D) &= 0.1 \quad P(T \mid D) = 0.95 \\
P(T \mid D^c) &= 0.2
\end{align*}
\]

\[
P(D \mid T) = \frac{P(D \cap T)}{P(T)} = \frac{P(D \cap T)}{P(D \cap T) + P(D^c \cap T)} = \frac{P(T \mid D) \cdot P(D)}{P(T \mid D) \cdot P(D) + P(T \mid D^c) \cdot P(D^c)} = \frac{0.95 \cdot 0.1}{0.95 \cdot 0.1 + 0.2 \cdot 0.9} = \frac{0.095}{0.095 + 0.18} = \frac{0.095}{0.28} = \frac{95}{285} = \frac{13}{55}
\]

\[
P(D) = 0.3
\]
\[
\begin{align*}
\frac{0.035}{0.035 + 0.18} &= \frac{95}{275} = \frac{13}{55} \\

\text{Note: } P(Q | T) &= \frac{P(T \mid D_1) \cdot P(D_1)}{P(T)} \\
&= \frac{P(T \mid D_1) \cdot P(D_1)}{P(T \cap D_1) + P(T \cap D_2) + \cdots + P(T \cap D_n)} \\
&= \frac{P(T \mid D_1) \cdot P(D_1)}{P(T \mid D_1) \cdot P(D_1) + P(T \mid D_2) \cdot P(D_2) + \cdots + P(T \mid D_n) \cdot P(D_n)} \\

D_1 \cup D_2 \cup \cdots \cup D_n &= \mathbb{S}, \quad \text{disjoint}\n\n\text{Let } n = 2, \quad D_1 \cup D_2 &= \mathbb{S} \\& \Longleftrightarrow \quad D \cup D^c &= \mathbb{S}

\text{Ex. 2 (2.13)} \quad \text{For each of the following, determine the constant } c \text{ so that } f(x) \text{ satisfies the condition of being a PMF of some RV } X. \\
\text{(a) } f(x) = \frac{x}{2}, \quad x = 1, 2, 3, 4 \quad \text{A) 7 \quad B) 3 \quad C) 4 \quad D) 6 \quad E) 10} \\
\text{where } x \in S_X = \{1, 2, 3, 4\} \\
\text{(b) } f(x) = c \cdot \frac{1}{4x}, \quad x = 1, 2, 3, -\infty \quad \text{A) 3 \quad B) \frac{1}{3} \quad C) 4 \quad D) \frac{1}{4} \quad E) 3} \quad \text{where } x \in S_X = \{1, 2, 3, 4, -\infty\} \\
\text{2 conditions: } f(x) \geq 0, \quad \sum_{x \in S_X} f(x) = 1
\]
(a) \( L = \sum_{x \in S_X} f(x) = \sum_{x=1}^{4} f(x) = \sum_{x=1}^{4} \frac{c}{x} = \frac{1}{c} + \frac{2}{c} + \frac{3}{c} + \frac{4}{c} = \frac{10}{c} \)

\[ c = 10 \]

(b) \( L = \sum_{x \in S_X} f(x) = \sum_{x=1}^{\infty} \frac{c}{x^2} = c \sum_{x=1}^{\infty} \frac{1}{x^2} \)

\[ = c \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) \]

\[ S = \frac{\pi^2}{6} \]

\[ \frac{S}{4} = \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \]

\[ \frac{S}{4} = \frac{1}{4} \hspace{1cm} 3S = 1 \hspace{1cm} S = \frac{1}{3} \]

\[ L = cS = c \cdot \frac{1}{3} \Rightarrow c = 3 \]

**Geometric series formula:** \( a + ar + ar^2 + \cdots = \frac{a}{1-r} \) \((|r| < 1)\)

\[ L = \sum_{x=1}^{\infty} \frac{c}{x^2} = \frac{c}{4} + \frac{c}{4^2} + \frac{c}{4^3} + \cdots = \frac{\frac{c}{4}}{1-\frac{1}{4}} = \frac{c}{3} \]

\[ a = \frac{c}{4} \hspace{1cm} n = \frac{1}{4} \]
Two independent dice are thrown. Let random variable \( (RV) X \) denote the sum of the numbers facing up. Find:

(a) \( \text{PDF of } X \) (draw bar graph)

(b) Compute \( E(X) \):

- \( A)7 \)
- \( B)7 \)
- \( C)8 \)
- \( D)9 \)
- \( E)14 \)

(c) \( M(t) = Mgf_X(t) \)

Denote the outcomes of two dice as \( D_1, D_2 \rightarrow X = D_1 + D_2 \)

What is sample space of \( X = \mathcal{S}_x = \{2, 3, \ldots, 12\} \)

\( \forall x \in \mathcal{S}_x, \quad f_X(x) := P(X = x) \)

\( f_X(2) = P(X = 2) = \frac{1}{36} \)

\( f_X(3) = P(X = 3) = \frac{2}{36} \)

\( f_X(4) = P(X = 4) = \frac{3}{36} \)

\( f_X(5) = P(X = 5) = \frac{4}{36} \)

\( f_X(6) = P(X = 6) = \frac{5}{36} \)

\( f_X(7) = P(X = 7) = \frac{6}{36} \)

\( f_X(8) = P(X = 8) = \frac{5}{36} \)

\( f_X(9) = \frac{4}{36} \)

\( f_X(10) = \frac{3}{36} \)

\( f_X(11) = \frac{2}{36} \)

\( f_X(12) = \frac{1}{36} \)
\[ f_x(12) = \mathbb{P}(X = 12) = f(51, 6, 6) = \frac{1}{76} \]

\[ f_x(2) \]

\[ \begin{array}{cccccccccc}
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline
\frac{5}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} & \frac{1}{76} \\
\end{array} \]

(6) \[ \mathbb{E}X = \sum_{x \in X} x \cdot f_x(x) \]

\[ = \sum_{x=2}^{12} x \cdot f_x(x) \]

\[ = 2 \cdot \frac{1}{76} + 3 \cdot \frac{2}{76} + \ldots + 7 \cdot \frac{6}{76} + 8 \cdot \frac{5}{76} + 9 \cdot \frac{4}{76} + 10 \cdot \frac{3}{76} + 11 \cdot \frac{2}{76} + 12 \cdot \frac{1}{76} \]

\[ = 7. \]

Usually, we say \[ \mathbb{E}X = \mathbb{E}(D_1 + D_2) = \mathbb{E}D_1 + \mathbb{E}D_2 = \frac{3}{2} - \frac{3}{2} = 2 \]

\[ \mathbb{E}D_1 = \sum_{x \in D_1} x \cdot f_a(x) = \sum_{x=1}^{6} x \cdot \frac{1}{6} = \frac{1}{6} + 2 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} = \frac{21}{6} = \frac{7}{2} \]