Discussion

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037401

(old Final Problem) let X_1, X_2, X_3 and X_4 be four independent Poisson RVs, which the rate of X_i is equal to $\lambda_i = i$. Define a new RV.

$$Y = 6 \cdot (X_1 + X_2 + X_3 + X_4)$$

(a) Compute mgf of Y.

(b) Compute $P(Y = 2019)$

(c) Compute $P(Y = 18)$ (How $Y/6$ is distributed ?)

Sol $X \sim Pois(\lambda)$ $M(t) = e^{\lambda(e^t - 1)}$

(a) $Y = 6(X_1 + X_2 + X_3 + X_4)$

$$M_Y(t) = E(e^{tY}) = E(e^{t \cdot 6(X_1+X_2+X_3+X_4)}) = E(e^{6tX_1} \cdot e^{6tX_2} \cdot e^{6tX_3} \cdot e^{6tX_4})$$

(Now using independence of X_1, X_2, X_3, X_4) $= \underline{E(e^{6tX_1})}\, \underline{E(e^{6tX_2})}\, E(e^{6tX_3})\, E(e^{6tX_4})$

$= M_{X_1}(6t)\, M_{X_2}(6t)\, M_{X_3}(6t)\, M_{X_4}(6t) =$

$$= e^{1(e^{6t}-1)} \cdot e^{2(e^{6t}-1)}\, e^{3(e^{6t}-1)}\, e^{4(e^{6t}-1)} = e^{10(e^{6t}-1)}$$

(b) $Y = 6\underbrace{(X_1 + X_2 + X_3 + X_4)}_{\text{integer}}$ $X_i \sim Pois(i)$ $X_i \in \{0, 1, 2, 3, \ldots\}$

$S_Y = \{0, 6, 12, \ldots\}$ = All nonnegative multiples of 6., but 2019 is not divisible by 6. $6 \nmid 2019$, which means $P(Y = 2019) = 0$

(c) $P(Y=18) = P(6(X_1+X_2+X_3+X_4) = 18) = P(X_1+X_2+X_3+X_4 = 3)$,

(c) $P(Y=18) = P(6(X_1+X_2+X_3+X_4)=18) = P(X_1+X_2+X_3+X_4=3)$

Say $X = X_1+X_2+X_3+X_4$

$$M_X(t) = E(e^{tX}) = E(e^{t(X_1+X_2+X_3+X_4)}) = E(e^{tX_1})\,E(e^{tX_2})\,E(e^{tX_3})\,E(e^{tX_4}) =$$

$$= \prod_{i=1}^{4} M_{X_i}(t) = \prod_{i=1}^{4} e^{i(e^t-1)} = e^{10(e^t-1)} = M_{Pois(10)}(t)$$

MGF of Poisson 10.

$X \sim Pois(10)$

$$P(Pois(10)=3) = \frac{10^3 e^{-10}}{3!} = \frac{500}{3e^{10}}$$

Note: Here, we actually showed $Pois(\lambda_1)+Pois(\lambda_2)+\cdots+Pois(\lambda_n) \sim Pois(\lambda_1+\lambda_2+\cdots+\lambda_n)$

(Assuming the independence.)

Ex2 (5.5.3) Let X equal to the widest diameter in mm of the fetal head measured in the 16^{th} and 25^{th} week of pregnancy. Assume that the distribution of X is $N(46.28, 40.96)$. Let $\bar{X}_{16}$ be the sample average of a random independent sample of 16 observation of X.

(a) Find $E(\bar{X}_{16})$ and $Var(\bar{X}_{16})$

(b) Find $P(44.42 \le \bar{X}_{16} \le 48.98)$

A) 0.16 B) 0.33 C) 0.51 D) 0.67 E) 0.84

sol) $$\bar{X}_{16} = \frac{X_1+X_2+\cdots+X_{16}}{16} = \frac{1}{16}X_1 + \frac{1}{16}X_2 + \cdots + \frac{1}{16}X_{16}$$

$$E(\bar{X}_{16}) = E\left(\frac{1}{16}X_1 + \frac{1}{16}X_2 + \cdots + \frac{1}{16}X_{16}\right)$$

$$= E(\tfrac{1}{16}X_1) + E(\tfrac{1}{16}X_2) + \cdots + E(\tfrac{1}{16}X_{16})$$

$$= \tfrac{1}{16}\underbrace{E(X_1)}_{46.28} + \tfrac{1}{16}E(X_2) + \cdots + \tfrac{1}{16}E(X_{16})$$

$X_i \sim N(46.28, 40.96)$

$$= 46.28 \qquad \left(E(\bar{X}_n) = EX_1 \right)$$

$$Var(\bar{X}_{16}) = Var\left(\tfrac{1}{16}X_1 + \tfrac{1}{16}X_2 + \cdots + \tfrac{1}{16}X_{16}\right)$$

$$= Var(\tfrac{1}{16}X_1) + Var(\tfrac{1}{16}X_2) + \cdots + Var(\tfrac{1}{16}X_{16})$$

(correct only when $Cov(X_i, X_j) = 0 \ \forall i \neq j$, but this is correct as $\{X_i\}$ are independent)

$$= \tfrac{1}{16^2}Var(X_1) + \tfrac{1}{16^2}Var(X_2) + \cdots + \tfrac{1}{16^2}Var(X_{16})$$

Note that

$$Var(c.X) = E((cX)^2) - (E(cX))^2$$

$$= E(c^2X^2) - (E\,cX)^2$$

$$c^2EX^2 - c^2EX^2 = c^2 Var(X^2)$$

$$= \tfrac{1}{16^2} \cdot 40.36 \cdot 16 = \frac{40.36}{16} = 2.56 = (1.6)^2$$

$$Var(\bar{X}_{16}) = \frac{40.36}{16} = 2.56$$

I can conclude $\bar{X}_{16} \sim N(46.28, 2.56) = \underbrace{46.28}_{\mu} + (1.6)Z$

$\hookrightarrow N(0,1)$

(b) $P(44.42 \leq \bar{X}_{16} \leq 48.98) = P(44.42 \leq 46.28 + 1.6\,Z \leq 48.98)$

$= P(-2.16 \leq 1.6\,Z \leq 2.4)$

$= P(-1.35 \leq Z \leq 1.5)$

$= \Phi(1.5) - \Phi(-1.35) = \overset{\sim 0.93}{\Phi(1.5)} - \left(1 - \overset{\sim 0.91}{\Phi(1.35)}\right)$

(look at standard table)

$= 0.84$