GALOIS REPRESENTATIONS MODULO p AND COHOMOLOGY OF HILBERT MODULAR VARIETIES

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The aim of this talk is to extend some arithmetic results on elliptic modular forms to the case of Hilbert modular forms. Among these results let's mention :

(i) the control of the image of the Galois representation modulo p [18][20][16],

(ii) Hida's congruence criterium outside an explicite set of primes p [11][10],

(iii) the freeness of the integral cohomology of the Hilbert modular variety over certain local components of the Hecke algebra and the Gorenstein property of these local algebras [12][8].

We study the arithmetic of the Hilbert modular forms by studying their mod p Galois representations and our main tool is the action of the inertia groups at p. In order to control this action, we compute the Hodge-Tate (resp. Fontaine-Laffaille) weights of the p-adique (resp. mod p) étale cohomology of the Hilbert modular variety. The cohomological part of our paper is inspired by the work of Mokrane, Polo and Tilouine [13, 15] on the cohomology of Siegel modular varieties and is built on the geometric constructions of [5] and [4].

Introduction

0.1. **Notations.** Let F be a totally real number field of degree d, of ring of integers $\mathfrak{o}$ and discriminant Δ_F. We put $\mathrm{N} = \mathrm{N}_{F/\mathbb{Q}}$. We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, by $\widetilde{F}$ the Galois closure of F in $\overline{\mathbb{Q}}$ and by J_F be the set of all embeddings of F in $\overline{\mathbb{Q}}$. The absolute Galois group of a field L is denoted by $\mathcal{G}_L$.

Let $k = \sum_{\tau\in J_F} k_\tau \tau \in \mathbb{Z}[J_F]$ be an algebraic weight. This means that all $k_\tau (\tau \in J_F)$ are greater or equal to 2 and of the same parity. We put then $k_0 = \max\{k_\tau | \tau \in J_F\}$, $m_\tau = (k_0 - k_\tau)/2$, $t = \sum_{\tau\in J_F} \tau$ and $n = k - 2t$.

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For each subset J of J_F we put $p(J)=\sum_{\tau\in J}(k_0-m_\tau-1)\tau+\sum_{\tau\in J_F\setminus J}m_\tau\tau$ and for each $a=\sum_{\tau\in J_F}a_\tau\tau\in\mathbb{Z}[J_F]$ we put $|a|=\sum_{\tau\in J_F}a_\tau\in\mathbb{Z}$.

We fix an ideal $\mathfrak{n}$ of $\mathfrak{o}$ and a Hecke character χ over F of type $-n_0t$ at infinity and whose conductor divides $\mathfrak{n}$. We denote by $S_k(\mathfrak{n},\chi)$ the corresponding space of Hilbert modular cuspforms.

Let $f\in S_k(\mathfrak{n},\chi)$ be a newform (= primitive, normalised eigenvector for the Hecke algebra $\mathbb{T}_k(\mathfrak{n},\chi)$) and let's denote for each prime ideal $\mathfrak{m}$ of $\mathfrak{o}$ by $a(f,\mathfrak{m})$ the eigenvalue of the standard Hecke operator $T_\mathfrak{m}\in\mathbb{T}_k(\mathfrak{n},\chi)$ on f.

Let $p\geq 5$ be a prime number. Let E be a p-adic field which is sufficiently large (this means containing $\widetilde{F}$ and the eigenvalues of the Hecke operators acting on $S_k(\mathfrak{n},\chi)$), of ring of integers $\mathcal{O}$, of maximal ideal $\mathcal{P}$ and of residue field κ.

By results of Taylor [21] and Blasuis-Rogawski [1] we can associate to f a continuous, absolutely irreducible, odd representation $\rho_f:\mathcal{G}_F\to\mathrm{GL}_2(E)$ which is unramified outside $\mathfrak{n}p$ and such that for each prime ideal v of $\mathfrak{o}$, not dividing $p\mathfrak{n}$, the following relations hold :

$$\mathrm{Tr}(\rho_f(\mathrm{Frob}_v))=a(f,v),\ \ \mathrm{Det}(\rho_f(\mathrm{Frob}_v))=\chi(v)\,\mathrm{N}(v)^{k_0-1}, \tag{1}$$

where Frob_v denotes the geometric Frobenius. We take the geometric Frobenius instead of the arithmetic one, in order to obtain positive integers for the Hodge-Tate weights of ρ_f.

By taking a Galois stable $\mathcal{O}$-lattice, we can define $\overline{\rho}_f=\rho_f \mod \mathcal{P}:\mathcal{G}_F\to\mathrm{GL}_2(\kappa)$, whose semi-simplification is independant of the particular choice of a lattice.

We consider also the p-adic étale cohomology groups of the Hilbert modular variety. By a result of Brylisnki and Labesse [2] the part cut in by the eigensystem associated to f is isomorphic to the tensor induced representation $\otimes\,\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_f$.

We consider the following the assumptions on p :

(I) p doesn't divide $\Delta=\Delta_F\,\mathrm{N}(\mathfrak{n})$,

(II) $p-1>\sum_{\tau\in J_F}(k_\tau-1)=|n|+d$.

Under the first assumption, the Hilbert modular variety has good reduction at p, admits smooth compactifications over $\mathbb{Z}_p$ (see [4]), and the Galois representations that we consider are crystalline.

The integer $|n|+d$ is equal to the difference $|p(J_F)|-|p(\varnothing)|$ between the biggest and the smallest Hodge-Tate weight of the Galois representation $\otimes\,\mathrm{Ind}_F^{\mathbb{Q}}\,\rho_f$. Under the second assumption, we are able to generalize results from the characteristic zero field case to the case of a p-adic ring or the case of a finite field of characteristic p. These two assumptions are also necessary to apply Fontaine-Laffaille's Theory [9], as well as Faltings' Comparison Theorem [6].

0.2. **Cohomological results.** Denote by $\mathfrak{M}$ the maximal ideal of the Hecke algebra $\mathbb{T} = \mathcal{O}[T_{\mathfrak{m}}\ ,\ \mathfrak{m} \subset \mathfrak{o}]$ acting on $S_k(\mathfrak{n}, \chi)$, corresponding to f and $\mathcal{P}$. Let Y be the Hilbert modular adelic variety of level $K_1(\mathfrak{n})$ and M be one of its connected components.

Theorem 1. *Under the assumption* (I)*, the Hodge-Tate weights of the crystalline representation* $\mathrm{H}^j(Y_{\overline{\mathbb{Q}}}, \mathbb{V}_n(\overline{\mathbb{Q}}_p))$ *belong to the set* $\{|p(J)|, J \subset J_F\ \ |J| \leq j\}$*. The multiplicity of* $|p(J)|$ *as a weight of* $\mathrm{H}^d(Y_{\overline{\mathbb{Q}}}, \mathbb{V}_n(\overline{\mathbb{Q}}_p))[f]$ *is equal to the cardinality of the set* $\{J' \subset J_F\ ,\ |p(J)| = |p(J')|\}$.

The proof relies on the Faltings' Comparison Theorem [6] relating the étale cohomology of M with coefficients in the local system $\mathbb{V}_n(\mathbb{Q}_p)$ to the De Rham logarithmic cohomology of the corresponding vector bundle $\overline{\mathcal{V}}_n$ over a smooth toroidal compactification $\overline{M}$ of M. Instead of usins Faltings' Comparison Theorem, one can use Tsuji's result for the étale cohomology with constant coefficients of the Kuga-Sato variety $\mathcal{A}^{|n|}$ ($|n|$-fold fiber product of the universal abelian variety $\mathcal{A}$ above the fine moduli space M^1 associated to M), whose toroidal compactifications have been constructed in [5]. The Hodge-Tate weights are given by the jumps of the Hodge filtration of the associated De Rham complex. These are computed, following [7], by using the Bernstein-Gelfand-Gelfand complex (BGG complex).

For some arithmetic applications we need to establish p-adic or mod p results analogous to the one above. To do this we adapt to the Hilbert modular case the techniques developped by Mokrane, Polo and Tilouine in the Siegel modular case. Namely, we construct an integral BGG complex for distribution algebras and then apply Faltings' Comparison Theorem mod p.

By using a "local-global" Galois argument, we obtain the following two theorems (the proof of the first one relies also on a theorem of Pink [14] on the étale cohomology of the restriction of a local system to the boundary of a Shimura variety).

Theorem 2. *Assume that the conditions* (I) *and* (II) *hold. Assume moreover that*

(Irr) $\overline{\rho}_f$ *is absolutely irreducible, and*

(MW) *the middle weight* $\frac{|p(J_F)|+|p(\varnothing)|}{2} = \frac{d(k_0-1)}{2}$ *does not belong to the set* $\{|p(J)|, J \subset J_F\}$.

Then

(i) *the* $\mathfrak{M}$*-local component of the boundary cohomology* $\mathrm{H}^\bullet_\partial(Y, \mathbb{V}_n(\kappa))_{\mathfrak{M}}$ *vanishes, and*

(ii) *the Poincaré pairing* $\mathrm{H}^d_!(Y, \mathbb{V}_n(\mathcal{O}))'_{\mathfrak{M}} \times \mathrm{H}^d_!(Y, \mathbb{V}_n(\mathcal{O}))'_{\mathfrak{M}} \to \mathcal{O}$ *is perfect (the notation* $'$ *means modulo torsion).*

Theorem 3. *Assume that the conditions* (I) *and* (II) *hold. Assume moreover that*
(LI) *the image of* $\mathrm{Ind}_F^{\mathbb{Q}}\overline{\rho}_f$ *is sufficiently large.*

Then
(i) $\mathrm{H}^\bullet(Y,\mathbb{V}_n(\kappa))_{\mathfrak{M}} = \mathrm{H}^d(Y,\mathbb{V}_n(\kappa))_{\mathfrak{M}}$,
(ii) $\mathrm{H}^\bullet(Y,\mathbb{V}_n(\mathcal{O}))_{\mathfrak{M}} = \mathrm{H}^d(Y,\mathbb{V}_n(\mathcal{O}))_{\mathfrak{M}}$ *is a free* $\mathcal{O}$*-module of finite rank and* $\mathrm{H}^\bullet(Y,\mathbb{V}_n(E/\mathcal{O})_{\mathfrak{M}} = \mathrm{H}^d(M,Y_k(E/\mathcal{O}))_{\mathfrak{M}}$ *is a divisible* $\mathcal{O}$*-module of finite corank.*

0.3. **Arithmetic results.** Let $\Lambda^*(\mathrm{Ad}^0(f),s)$ be the imprimitive adjointe L-function of f, completed by its Euler factors at infinity and let $W(f)$ be the complex constant from the functional equation of the standard L-function of f. We denote by $\Omega_f^\pm \in \mathbb{C}^\times/\mathcal{O}^\times$ the periods defined by the Eichler-Shimura-Harder isomorphism.

Theorem A *Let* f *and* p *be such that the conditions* (I), (II) , (Irr) *and* (MW) *hold.*
Then, if $\mathcal{P}$ *divides* $\frac{W(f)\Lambda^*(\mathrm{Ad}^0(f),1)}{\Omega_f^+\Omega_f^-}$, *then it exists another normalised eigenform* $g \in S_k(\mathfrak{n},\chi)$ *such that* $f \equiv g \pmod{\mathcal{P}}$, *in the sense that the corresponding eigenvalues of* f *and* g *are congruent modulo* $\mathcal{P}$.

If $\mathfrak{n} = \mathfrak{o}$ and $k = 2t$ or if F is a quadratic real field, k is parallel and f is ordinary at p, Ghate [10] has obtained an analogous result to the one above. The proof follows closely the original one given by Hida [11] in the elliptic modular case, and uses the theorem 2, as well as a formula of Shimura [19] relating $\frac{W(f)\Lambda^*(\mathrm{Ad}^0(f),1)}{\Omega_f^+\Omega_f^-}$ to the number of cohomological congruences.

Theorem B *Let* f *and* p *be such that the conditions* (I), (II) *and* (LI) *hold. Then*
(i) $\mathrm{H}^\bullet(Y,\mathbb{V}_n(\kappa))[\mathfrak{M}]$ *is* κ*-vector space of dimension* 2^d.
(ii) $\mathrm{H}^\bullet(Y,\mathbb{V}_n(\mathcal{O}))_{\mathfrak{M}}$ *is free of rank* 2^d *over* $\mathbb{T}_{\mathfrak{M}}$.
(iii) $\mathbb{T}_{\mathfrak{M}}$ *is Gorenstein.*

The proof of (i) relies on the theorem 3 and the q-expansion principle (see [8], [3]). Then (ii) and (iii) are deduces as in [12].

Remark 4. In theorems A and B do not use any of the followins assumptions : f ordinary at p, parallel weight, trivial strict class group of F, level one, nor ρ_f minimal.

0.4. **Galois results.** In this section we study the images of the Galois representations $\overline{\rho}_f$ and $\mathrm{Ind}_F^{\mathbb{Q}}\overline{\rho}_f$, in order to prove that conditions (Irr) and (LI) are satisfied in most cases, and give explicite corollaries of the theorems A and B. This is achieved by generalizing to the Hilbert modular case theorems of Serre [18] and Ribet [16] on elliptic modular forms.

Denote by $\widehat{F}$ the compositum of $\widetilde{F}$ and the subfield of $\overline{\mathbb{Q}}$ fixed by $\{g \in \mathcal{G}_{\mathbb{Q}}, \mathrm{Det}(\overline{\rho}_f(g)) \in \mathbb{F}_p^{\times}\}$.

Proposition 5. *Assume that the conditions* (I) *and* (II) *hold.*

(i) *If for all $J \subset J_F$, there exists an unit $\epsilon \equiv 1 \pmod{\mathfrak{n}}$ such that p does not divide the integer $\mathrm{N}(\epsilon^{p(J)}-1)$ (in particular the weight k should not be parallel), then* (**Irr**) *holds.*

(ii) *Assume* (**Irr**), $d(p-1) > 5(\sum_{\tau \in J_F}(k_\tau - 1))$ *and that the following condition is satisfied :*

(**not−CM**) *for any quadratic CM-extension M of F, of discriminant dividing $\mathfrak{n}$, and in which all the primes $\mathfrak{p}$ of F above p split, it does not exist a Hecke character ψ over M of type $\widetilde{k} = (k, (k_0-1)t-k)$ at infinity and of conductor $\mathfrak{n}\Delta_{M/F}^{-1}$ such that $f \equiv \mathrm{Ind}_M^F \psi \pmod{\mathcal{P}}$.*

Then, there exists a power q of p such that we have $\mathrm{Im}(\overline{\rho}_f|_{\mathcal{G}_{\widehat{F}}}) = \mathrm{GL}_2(\mathbb{F}_q)^{\mathcal{D}}$ or $(\mathbb{F}_{q^2}^{\times}\,\mathrm{GL}_2(\mathbb{F}_q))^{\mathcal{D}}$. In this case, we say that the image of $\overline{\rho}_f$ is large.

(iii) *Assume, either that the image of $\overline{\rho}_f$ is large and k is not induced, either that f is a theta series coming from a CM field M/F whose conductor $\Delta_{M/F}$ is primitive. Then* (**LI**) *holds.*

Remark 6. (i) As a corollary we obtain that if the conditions (I) and (II) hold and if the form f is not a theta series then the image of $\overline{\rho}_f$ is large for all than finitely many primes p. Moreover, if the the weight k is not parallel, then we get an explicite bound for these p's. In the case where k is parallel ($k = k_0 t$), one could try to generalize the crystalline methods of Faltings and Jordan [8], in order to prove (Irr) under the condition that p does not divide the constant term of an Eisenstein series of weight $k_0 t$, in other words the value at $1-k_0$ of the L-function of a Hecke character over F of finite order.

(ii) the (not-CM) condition is used to prove that the projective image of $\overline{\rho}_f$ is not dihedral. The primes p for which the congruence $f \equiv \mathrm{Ind}_M^F \psi \pmod{\mathcal{P}}$ may appear should be controled by the special value of the L-function associated to the character CM $\psi\psi^c$ (In the case elliptic this is proved by H. Hida [11] and K. Ribet [17]; see also Theorems A and B above).

(iii) the condition $d(p-1) > 5\sum_{\tau \in J_F}(k_\tau - 1)$ is used to rule out the exceptional case where the projective image of $\overline{\rho}_f$ is isomorphic to one of the groups A_4, S_4 or A_5.

References

[1] D. Blasius and J. Rogawski, *Motives for Hilbert Modular Forms*, Invent. Math., 114 (1993).

[2] J.-L. Brylinski and J.-P. Labesse, *Cohomologies d'Intersection et Fonctions L de Certaines Variétés de Shimura*, Ann. Sci. Éc. Norm. Sup., 17 (1984), pp. 361–412.

[3] F. Diamond, *On the Hecke action on the cohomology of Hilbert-Blumenthal surfaces* , Contemporary Math., 210 (1998).

[4] M. Dimitrov, *Compactifications arithmétiques des variétés de Hilbert et formes modulaires de Hilbert pour* $\Gamma_1(\mathfrak{c},\mathfrak{n})$, in Geometric Aspects of Dwork Theory, A Volume in memory of Bernard Dwork, A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, and F. Loeser, eds., Walter de Gruyter, 2003.

[5] M. Dimitrov and J. Tilouine, *Variétés et formes modulaires de Hilbert arithmétiques pour* $\Gamma_1(\mathfrak{c},\mathfrak{n})$, in Geometric Aspects of Dwork Theory, A Volume in memory of Bernard Dwork, A. Adolphson, F. Baldassarri, P. Berthelot, N. Katz, and F. Loeser, eds., Walter de Gruyter, 2003.

[6] G. Faltings, *Crystalline cohomology and p-adic Galois representations*, in Algebraic analysis, geometry, and number theory, J. H. U. Press, ed., 1989, pp. 25–80.

[7] G. Faltings and C.-L. Chai, Degeneration of Abelian Varieties, Springer-Verlag, 1990.

[8] G. Faltings and B. Jordan, *Crystalline cohomology and* $\mathrm{GL}(2,\mathbb{Q})$, Israel J. Math., 90 (1995), pp. 1–66.

[9] J.-M. Fontaine and G. Laffaille, *Construction de représentations p-adiques*, Ann. Sci. ENS, 15 (1982), pp. 547–608.

[10] E. Ghate, *Adjoint L-values and primes of congruence for Hilbert modular forms*, Compositio Math., 132 (2002), pp. 243–281.

[11] H. Hida, *Congruences of Cusp Forms and Special Values of Their Zeta Functions*, Inventiones Math., 63 (1981), pp. 225–261.

[12] B. Mazur, *Modular curves and the Eisenstein ideal*, Inst. Hautes Études Sci. Publ. Math., 47 (1977), pp. 33–186.

[13] A. Mokrane and J. Tilouine, *Cohomology of Siegel varieties with p-adic integral coefficients and Applications,* in Cohomology of Siegel Varieties, Astérisque, 280 (2002), pp. 1–95.

[14] R. Pink, *On l-adic sheaves on Shimura varieties and their higher images in the Baily-Borel compactification*, Math. Ann., 292 (1992), pp. 197–240.

[15] P. Polo and J. Tilouine, *Bernstein-Gelfand-Gelfand complexes and cohomology of nilpotent groups*, Astérisque, 280 (2002).

[16] K. Ribet, *On l-Adic Representations Attached to Modular Forms*, Invent. Math., 28 (1975).

[17] ———, *Mod p Hecke operators and congruences between modular forms*, Invent. Math., 71 (1983), pp. 193–205.

[18] J.-P. Serre, *Propriétés galoisinnes des points d'ordre fini des courbes elliptiques*, Invent. Math., 15 (1972).

[19] G. Shimura, *The Special Values of the Zeta Functions Associated with Hilbert Modular Forms*, Duke Math. J., 45 (1978), pp. 637–679.

[20] H. Swinnerton-Dyer, *On l-adic representations and congruences for coefficients of modular forms*, in International Summer School on Modular Forms, 1972, vol. 350 of LNM, 1973.

[21] R. Taylor, *On Galois Representations Associated to Hilbert Modular Forms*, Inventiones Math., 98 (1989).

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