ON THE MODULARITY OF WILDLY RAMIFIED GALOIS REPRESENTATIONS

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1. Introduction

There is considerable interest in continuous homomorphisms

$$\rho : G_{\mathbb{Q}} \to GL_2\left(\overline{\mathbb{Q}}_\ell\right)$$

where $G_{\mathbb{Q}} = \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right)$ is the absolute Galois group of $\mathbb{Q}$ and ℓ is a fixed rational prime. For example, $\rho = \rho_{E,\ell}$ may be the ℓ-adic representation of an elliptic curve E over $\mathbb{Q}$, or $\rho = \rho_f$ may be the ℓ-adic representation associated to a modular form. The continuity of such Galois representations implies the image lies in $GL_2(\mathcal{O})$ for some ring of integers $\mathcal{O}$ with maximal ideal λ in a finite extension K of $\mathbb{Q}_\ell$; then $k = \mathcal{O}/\lambda$ is a finite extension of $\mathbb{F}_\ell$. We define the residual representation $\overline{\rho}$ as the composition

$$\overline{\rho} : G_{\mathbb{Q}} \to GL_2\left(\mathcal{O}\right) \to GL_2\left(\overline{\mathbb{F}}_\ell\right).$$

For example, $\overline{\rho} = \overline{\rho}_{E,\ell}$ may be the mod ℓ representation of an elliptic curve, or $\overline{\rho} = \overline{\rho}_f$ may be the mod ℓ reduction of the representation associated to a modular form. We also consider the projective representation $\widetilde{\rho}$ as the composition

$$\widetilde{\rho} : G_{\mathbb{Q}} \to GL_2\left(\overline{\mathbb{Q}}_\ell\right) \to PGL_2\left(\overline{\mathbb{Q}}_\ell\right).$$

A common question is: Given a continuous ℓ-adic Galois representation ρ such that $\overline{\rho}$ is modular i.e. $\overline{\rho} \simeq \overline{\rho}_f$, when is ρ modular i.e. $\rho \simeq \rho_f$?

Our main result is:

Theorem 1. *For ℓ an odd prime, let $\rho : G_{\mathbb{Q}} \to GL_2(\mathcal{O})$ be a continuous ℓ-adic representation such that*

1. *ρ is ordinary and ramified at finitely many primes;*
2. *$\overline{\rho}$ is absolutely irreducible when restricted to Gal$\left(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\,\ell})\right)$, modular, and wildly ramified at ℓ.*

Then ρ is ℓ-adically modular i.e. $\rho \simeq \rho_f$ for an ℓ-adic cusp form f.

One application is:

Theorem 2. *For ℓ an odd prime, let $\rho : G_{\mathbb{Q}} \to GL_2\left(\mathcal{O}\right)$ be a continuous Galois representation such that*

1. *ρ is ramified at finitely many primes;*
2. *$\overline{\rho}$ is absolutely irreducible when restricted to Gal$\left(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{(-1)^{(\ell-1)/2}\,\ell})\right)$, modular, and wildly ramified at ℓ;*
3. *$\rho(G_\ell)$ is finite and $\widetilde{\rho}(G_\ell)$ is a cyclic group of ℓ-power order.*

Then $\imath \circ \rho : G_{\mathbb{Q}} \to GL_2\left(\mathbb{C}\right)$ is modular for each embedding $\imath : K \hookrightarrow \mathbb{C}$.

For any continuous *complex* Galois representation ρ, there is a finite extension $L/\mathbb{Q}$ which makes the following diagram commute:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Gal}\left(\overline{\mathbb{Q}}/L\right) & \longrightarrow & \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) & \longrightarrow & \mathrm{Gal}\left(L/\mathbb{Q}\right) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \widetilde{\rho}} & & \\ 1 & \longrightarrow & \mathbb{C}^\times & \longrightarrow & GL_2(\mathbb{C}) & \longrightarrow & PGL_2(\mathbb{C}) & \longrightarrow & 1 \end{array}$$

The projective image $\widetilde{\rho}(G_\mathbb{Q})$ can either be cyclic, dihedral, tetrahedral ($A_4 \simeq PSL_2(\mathbb{F}_3)$), octahedral ($S_4 \simeq PGL_2(\mathbb{F}_3)$), or icosahedral ($A_5 \simeq PSL_2(\mathbb{F}_5)$). Our third result is:

Theorem 3. *Let $\rho : G_\mathbb{Q} \to GL_2(\mathbb{C})$ be a continuous representation with non-solvable image. If L is the splitting field of a quintic $x^5 + B\,x + C$ such that $75\,C^2/\sqrt{256\,B^5 + 3125\,C^4}$ is the square of a 5-adic unit, then ρ is (classically) modular.*

Here is one example: the polynomials

$$x^5 + 10\,x^3 + 10\,x^2 + 35\,x + 18 \qquad \text{and} \qquad 5\,x^5 + 20\,x + 16$$

generate the same splitting field. An infinite family of examples are

$$x^5 + 5\left(\frac{9 - 5\,u^4}{5\,u^4}\right)x + 4\left(\frac{9 - 5\,u^4}{5\,u^4}\right), \qquad u \in \mathbb{Q} \cap \mathbb{Z}_5^\times .$$

This result gives the first known proof of infinitely many examples of icosahedral Galois representations satisfying Artin's Conjecture which are ramified at 5.

2. Applications of Theorem 1

We explain how the second main result follows from the first. We know from Theorem 1 that, under suitable hypotheses, ρ is ℓ-adically modular, but this cusp form may not be classical.

Proof of Theorem 2. If $\rho(G_\ell)$ is finite and $\widetilde{\rho}(G_\ell)$ is cyclic of ℓ-power order, there exist characters χ_1 and χ_2 such that the characters $\chi_\ell = {\chi_1}^{-1}\chi_2$ and $\chi_0 = \det\rho/(\chi_1\,\chi_2)$ are wildly ramified at ℓ and unramified at ℓ, respectively. Consider the twists $\rho_i = \chi_i^{-1} \otimes \rho$ for $i = 1, 2$; then

$$\rho_1|_{I_\ell} \simeq \begin{pmatrix} \chi_\ell & * \\ & 1 \end{pmatrix} \quad \text{and} \quad \rho_2|_{I_\ell} \simeq \begin{pmatrix} {\chi_\ell}^{-1} & * \\ & 1 \end{pmatrix}.$$

Each ρ_i is ordinary and residually modular, so using Theorem 1 let $f(\tau) = \sum_n a_n\,q^n$ denote the ℓ-adic form associated with ρ_1 and $g(\tau) = \sum_n b_n\,q^n$ denote the ℓ-adic form associated with ρ_2. We have

(1) f and g are ordinary cusp forms of weight 1;
(2) f and g have nebentype $\det\rho_1 = \chi_0 \cdot \chi_\ell$ and $\det\rho_2 = \chi_0 \cdot {\chi_\ell}^{-1}$;
(3) $\rho_1 = \chi_\ell \otimes \rho_2$ i.e. $a_p = \chi_\ell\,(\mathrm{Frob}_p) \cdot b_p$ for almost all $p \neq \ell$; and
(4) $a_\ell = b_\ell = \chi_0\,(\mathrm{Frob}_\ell)$.

We use Kevin Buzzard's results to "glue" them together. □

Now we explain how the new case of Artin's conjecture follows from the second main result.

Proposition 4. *Let $q(x) = x^5 + B\,x + C$ be a quintic over $\mathbb{Q}$ with Galois group A_5, and denote L as its splitting field. For $t \in \mathbb{Q}^\times$, define the quintic and the curve*

$$q_t(x) = x^5 + 5\left(\frac{9 - 5\,t^2}{5\,t^2}\right) x + 4\left(\frac{9 - 5\,t^2}{5\,t^2}\right); \qquad E_t:\ y^2 = x^3 + 2\,x^2 + \frac{3 + \sqrt{5}\,t}{2\sqrt{5}\,t}.$$

We have the following.

(1) *E_t is a 2-isogenous $\mathbb{Q}$-curve. If L_t denotes the splitting field of $q_t(x)$, then $L_t(\sqrt{5}) \subseteq \mathbb{Q}(E[5])$. Specifically, $L_t(\sqrt{5})$ is the field generated by sum $x_P + x_{2P}$ of x-coordinates of the 5-torsion of E.*

(2) *$Gal(L_t/\mathbb{Q}) \subseteq A_5$. If t is square of a rational number then $Gal(L_t/\mathbb{Q}) = A_5$. If t is the square of a 5-adic unit, then the decomposition, inertia, and wild inertia groups at 5 are cyclic of order 5.*

(3) *When $t = 75\,C^2/\sqrt{Disc(q)}$ then $L = L_t$.*

(4) *There exists $\omega : Gal\left(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt{5})\right) \to \mathbb{C}^\times$ such that $\rho_E^{(1)} = \omega \otimes (\pi \circ \overline{\rho}_{E,5})$ and $\rho_E^{(2)} = \omega \otimes \rho_{E,5}$ are restictions of representations of $G_\mathbb{Q}$.*

(5) *E_t is modular. In particular, $\rho_E^{(2)}$ is modular while $\rho_E^{(1)}$ is residually modular.*

Any A_5-extension which is unramified outside of $\{2, 5, \infty\}$ comes from such quintics. Any quintic in Bring-Jerrard form yields a modular residual representation.

Proof of Theorem 3: Set $t = 75\,C^2/\sqrt{256\,B^5 + 3125\,C^4}$ and $E = E_t$. Then ρ is a twist of $\rho_E^{(1)}$ because there are only two projective complex representations of A_5 and they are Galois conjugates of each other. The result follows from Theorem 2. $\square$

3. Proof of Theorem 1

3.1. **Universal Deformation Ring.** The residual representation $\overline{\rho}$ induces an action of the absolute Galois group on the k-vector space of 2×2 matrices with trace zero given by $\sigma \cdot m = \overline{\rho}(\sigma)\ m\ \overline{\rho}(\sigma)^{-1}$; we denote this k-vector space with such an action by $\mathrm{ad}^0\overline{\rho}$.

Proposition 5. *Let $\overline{\rho} : G_\mathbb{Q} \to GL_2(k)$ be a continuous Galois representation, and let ϵ be an infinitesimal i.e. $\epsilon^2 = 0$. The equivalence classes of infinitesimal deformations $\overline{\rho}_\epsilon : G_\mathbb{Q} \to GL_2\left(k[\epsilon]\right)$ satisfying $\overline{\rho}_\epsilon \equiv \overline{\rho} \mod \epsilon\, k[\epsilon]$ and $\det \overline{\rho}_\epsilon = \det \overline{\rho}$ are in one-to-one correspondence with the cohomology classes in $H^1\left(G_\mathbb{Q},\ ad^0\overline{\rho}\right)$.*

Proof. Express an infinitesimal deformation in the form $\overline{\rho}_\epsilon(\sigma) = (1_2 + \epsilon\,\xi_\sigma)\,\overline{\rho}(\sigma)$ for some $\xi_\sigma \in \mathrm{Mat}_2(k)$, where ξ_σ must have trace zero since $\det \overline{\rho}(\sigma) = \det \overline{\rho}_\epsilon(\sigma) = (1 + \epsilon\,\mathrm{tr}\,\xi_\sigma)\det \overline{\rho}(\sigma)$. Equivalence classes of homomorphisms are in one-to-one correspondence with $\xi \in H^1\left(G_\mathbb{Q},\ \mathrm{ad}^0\overline{\rho}\right)$. $\square$

Recall that for each place ν of $\mathbb{Q}$, we have restriction maps $\mathrm{res}_\nu : H^1\left(G_\mathbb{Q},\ \mathrm{ad}^0\overline{\rho}\right) \to H^1\left(G_\nu,\ \mathrm{ad}^0\overline{\rho}\right)$. When $\nu \neq \ell$ each deformation $\overline{\rho}_\epsilon$ should be (un)ramified when $\overline{\rho}$ is (un)ramified so the restriction of a class from $H^1\left(G_\nu,\ \mathrm{ad}^0\overline{\rho}\right)$ to $H^1\left(I_\nu,\ \mathrm{ad}^0\overline{\rho}\right)$ should be trivial. Define

$$H^1_f\left(G_\nu,\ \mathrm{ad}^0\overline{\rho}\right) = \ker\left[H^1\left(G_\nu,\ \mathrm{ad}^0\overline{\rho}\right) \to H^1\left(I_\nu,\ \mathrm{ad}^0\overline{\rho}\right)\right] \qquad \text{for } \nu \neq \ell.$$

When $\nu = \ell$ and $\overline{\rho}$ is ordinary i.e. the restriction of $\overline{\rho}$ to G_ℓ is upper-triangular, each deformation $\overline{\rho}_\epsilon$ should be ordinary as well. We choose

$$\begin{aligned} H^1_f\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) &\subseteq \ker\left[H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) \to H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}/\mathrm{ad}^1\overline{\rho}\right)\right] \\ &= \mathrm{im}\left[H^1\left(G_\ell,\, \mathrm{ad}^1\overline{\rho}\right) \to H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right)\right]. \end{aligned}$$

When restricted to the inertia group, the diagonal terms of $\overline{\rho}_\epsilon$ should be the same as those of $\overline{\rho}$ so the restriction of a class from $H^1\left(G_\ell,\, \mathrm{ad}^1\overline{\rho}\right)$ to $H^1\left(I_\ell,\, \mathrm{ad}^1\overline{\rho}/\mathrm{ad}^2\overline{\rho}\right)$ should be trivial. Define

$$\begin{aligned} &H^1_f\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) \\ &= \mathrm{im}\Big[\ker\left[H^1\left(G_\ell,\, \mathrm{ad}^1\overline{\rho}\right) \to H^1\left(I_\ell,\, \mathrm{ad}^1\overline{\rho}/\mathrm{ad}^2\overline{\rho}\right)\right] \to H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right)\Big]. \end{aligned} \tag{1}$$

Remark. We explain how this definition for $\nu = \ell$ compares to the usual one. In general we have the exact sequence

$$H^1_f\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) \longrightarrow H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) \longrightarrow H^1\left(I_\ell,\, \mathrm{ad}^0\overline{\rho}/\mathrm{ad}^2\overline{\rho}\right)$$

but the map $H^1\left(G_\ell,\, \mathrm{ad}^1\overline{\rho}\right) \to H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right)$ is not an injection. However when $\det\rho = \varepsilon_\ell$ is the cyclotomic character, the group $H^0\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}/\mathrm{ad}^1\overline{\rho}\right)$ is trivial, so we recover the usual definition:

$$H^1_f\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) = \ker\left[H^1\left(G_\ell,\, \mathrm{ad}^0\overline{\rho}\right) \to H^1\left(I_\ell,\, \mathrm{ad}^0\overline{\rho}/\mathrm{ad}^2\overline{\rho}\right)\right].$$

Fix a finite set Σ of places that does not contain ℓ. We define the Selmer group $H^1_\Sigma\left(\mathbb{Q},\, \mathrm{ad}^0\overline{\rho}\right)$ as the the collection of classes $\xi \in H^1\left(G_\mathbb{Q},\, \mathrm{ad}^0\overline{\rho}\right)$ such that $\mathrm{res}_\nu(\xi) \in H^1_f\left(G_\nu,\, \mathrm{ad}^0\overline{\rho}\right)$ for all places $\nu \notin \Sigma$. We say a representation $\rho' : G_\mathbb{Q} \to GL_2(\mathcal{O}')$ is a deformation of $\overline{\rho}$ of type Σ if

(1) $\overline{\rho}' \simeq \overline{\rho} \otimes_k k'$ and $\det\rho' = \det\rho$;
(2) $\rho'|_{I_\nu} \simeq \rho|_{I_\nu} \otimes_K K'$ for all $\nu \notin \Sigma$; and
(3) $\rho'|_{G_\ell} \simeq \begin{pmatrix} \chi_\ell & * \\ & \chi_0 \end{pmatrix}$ where $\chi_\ell = \det\rho \cdot {\chi_0}^{-1}$ and $\chi_0|_{I_\ell} = 1$.

Proposition 6. (1) *There exists a universal deformation* $\rho^{univ}_\Sigma : G_\mathbb{Q} \to GL_2(R_\Sigma)$ *of* $\overline{\rho}$ *of type* Σ.
(2) R_Σ *can be topologically generated as an* $\mathcal{O}$*-algebra by* $\dim_k H^1_\Sigma\left(\mathbb{Q},\, ad^0\overline{\rho}\right)$ *elements.*
(3) *We have the identity*

$$\dim_k H^1_f\left(G_\ell,\, ad^0\overline{\rho}\right) = 1 + \dim_k H^0\left(G_\ell,\, ad^0\overline{\rho}\right).$$

3.2. **Modular Deformation Ring.** Fix a positive integer κ, a positive integer $N = N_0\,\ell$ in terms of an integer N_0 prime to ℓ, and a Dirichlet character $\chi : (\mathbb{Z}/N\,\mathbb{Z})^\times \to \mathbb{C}^\times$.

Denote $\Lambda = \mathcal{O}[[X]]$ as the power series ring in the variable X. For each positive rational integer κ, we have a specialization map $\varphi_\kappa : \Lambda \to \mathcal{O}$ defined by $1 + X \mapsto (1+\ell)^\kappa$. Consider a collection of forms $\sum_n a_n^{(\kappa)}\, q^n \in S_\kappa(N, \chi\,\omega_\ell^{1-\kappa})$ for each κ. We call such a collection a Hida family if for each n there exist power series $a_n(X) \in \Lambda$ such that $a_n^{(\kappa)} = a_n\left((1+\ell)^\kappa - 1\right)$ for all but finitely many κ. The collection $S(N,\chi)$ of formal series $F(X;\tau) = \sum_n a_n(X)\, q^n$ are called Λ-adic cusp forms of level N and nebentype χ if $F\left((1+\ell)^\kappa - 1;\, \tau\right) \in S_\kappa\left(N, \chi\,\omega_\ell^{1-\kappa}\right)$ for all but finitely many κ. For all κ, the specializations $\varphi_\kappa \circ F$ are called ℓ-adic modular forms – even when they are not classical.

Define the Hecke operators

$$(F|T_p)(X;\,\tau) = \sum_{p|n} a_n(X)\, q^{n/p} + \chi(p)\,\sigma_\ell(p) \sum_n a_n(X)\, q^{pn} \qquad \text{for } p \nmid N \text{ and}$$
$$(F|U_p)(X;\,\tau) = \sum_{p|n} a_n(X)\, q^{n/p} \qquad \text{for } p \mid N,$$

in terms of the character $\sigma_\ell : \mathbb{Z}_\ell^\times \to \Lambda^\times$ mapping $d \mapsto ((1+X)/(1+\ell))^{s(d)}$ where $s(d) = \log\langle d\rangle / \log(1+\ell) \in \mathbb{Z}_\ell$; and define the Λ-adic Hecke algebra $h(N,\chi)$ as the Λ-algebra generated by these operators. One checks that the composition $\varphi_\kappa \circ \sigma_\ell$ sends $p \mapsto \langle p\rangle^{\kappa-1} = (\varepsilon_\ell/\omega_\ell)^{\kappa-1}$ for $p \nmid N$, and for any operator T and Λ-adic cusp form F we have $(\varphi_\kappa \circ F)|T = \varphi_\kappa \circ (F|T)$.

Denote $h^0(N,\chi) = e \cdot h(N,\chi)$ as the ordinary part of the Hecke algebra, and define the ordinary Λ-adic cusp forms $S^0(N,\chi)$ as those cusp forms.

Associated to each ordinary normalized Λ-adic eigenform $F(X;\tau) = \sum_n a_n(X)\, q^n$ of level N and nebentype χ there is a continuous Λ-adic Galois representation $\rho_F : G_\mathbb{Q} \to GL_2\left(\overline{\mathbb{Q}}_\ell[[X]]\right)$ with the properties

1. ρ_F is unramified outside of the primes that divide N;
2. $\operatorname{tr}\rho_F(\mathrm{Frob}_p) = a_p(X)$ for $p \nmid N$;
3. $\det \rho_F = \chi \cdot \sigma_\ell$; and
4. ρ_F is ordinary.

As F specializes to ℓ-adic cusp forms $f = \varphi_\kappa \circ F$ the Galois representation ρ_F specializes to ℓ-adic representations $\rho_f = \varphi_\kappa \circ \rho_F$.

Define a map $\pi : h^0(N_\Sigma,\chi) \to \prod_{F \text{ type } \Sigma} \Lambda$ by

$$\pi: \quad T_p \mapsto (\ldots, \pi_F(T_p), \ldots) = (\ldots, \operatorname{tr}\rho_F(\mathrm{Frob}_p), \ldots) \quad \text{for} \quad p \nmid N_\Sigma;$$

where each component corresponds to a cusp form for $\overline{\rho}$ of type Σ. We define the modular deformation ring $\mathbb{T}_\Sigma$ to be the Λ-algebra generated by the images of T_p for $p \nmid N_\Sigma$. Note there is a continuous representation

$$\rho_\Sigma^{\mathrm{mod}} : G_\mathbb{Q} \to GL_2(\mathbb{T}_\Sigma)$$

where $\rho_\Sigma^{\mathrm{mod}} \simeq \prod_{F \text{ type } \Sigma} \rho_F$ is a deformation of $\overline{\rho}$ of type Σ.

3.3. **Isomorphism Criteria.** The modular deformation ring $\mathbb{T}_\Sigma$ is a complete, Noetherian, local Λ-algebra so there is a unique Λ-algebra surjection $\phi_\Sigma : R_\Sigma \to \mathbb{T}_\Sigma$ of the universal deformation ring such that $\rho_\Sigma^{\mathrm{mod}} \simeq \phi_\Sigma \circ \rho_\Sigma^{\mathrm{univ}}$. The following result states that this map is an isomorphism if and only if it is an isomorphism upon specializing the weight.

Proposition 7. *TFAE:*

1. $\phi_\Sigma : R_\Sigma \to \mathbb{T}_\Sigma$ *is an isomorphism.*
2. $\phi_\Sigma^{(\kappa)} : R_\Sigma^{(\kappa)} \to \mathbb{T}_\Sigma^{(\kappa)}$ *is an isomorphism for all positive integers* κ.
3. $\phi_\Sigma^{(\kappa)} : R_\Sigma^{(\kappa)} \to \mathbb{T}_\Sigma^{(\kappa)}$ *is an isomorphism for some positive integer* κ.

If any of the above hold for every finite set Σ *not containing* ℓ, *then* ρ *is* ℓ-*adically modular i.e.* $\rho \simeq \rho_f$ *for some* ℓ-*adic cusp form* f.

Proof of Theorem 1. It suffices to show $\phi_\Sigma^{(2)} : R_\Sigma^{(2)} \to \mathbb{T}_\Sigma^{(2)}$ (i.e. the weight $\kappa = 2$ case). The following commutative diagram is exact for the unique surjection ϕ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker\phi & \longrightarrow & R_\Sigma^{(2)} & \xrightarrow{\phi} & \mathcal{O} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow \phi_\Sigma^{(2)} & & \big\downarrow & & \\ 0 & \longrightarrow & \ker\pi_f & \longrightarrow & \mathbb{T}_\Sigma^{(2)} & \xrightarrow{\pi_f} & \mathcal{O} & \longrightarrow & 0 \end{array}$$

Denote the "tangent space" of $R_\Sigma^{(2)}$ as $\Phi_\Sigma = (\ker\phi) \, / \, (\ker\phi)^2$, as well as the ideals $\mathfrak{p}_\Sigma = \ker\pi_f$ and $I_\Sigma = \mathrm{Ann}_{\mathbb{T}_\Sigma^{(2)}} \ker\pi_f$ in $\mathbb{T}_\Sigma^{(2)}$. We assume for the moment that there exists a family of $\mathbb{T}_\Sigma^{(2)}$-modules J_Σ satisfying the following properties:

HM1: J_Σ is free over $\mathcal{O}$ with $\mathrm{rank}_{\mathcal{O}} J_\Sigma = 2 \cdot \mathrm{rank}_{\mathcal{O}} \mathbb{T}_\Sigma^{(2)}$.
HM2: $\mathrm{rank}_{\mathcal{O}} J_\Sigma[\mathfrak{p}_\Sigma] = 2$.

Step 1: The Minimal Case. $R_\emptyset^{(2)} \to \mathbb{T}_\emptyset^{(2)}$ is an isomorphism of complete intersections if and only if $\overline{R}_\emptyset^{(2)} \to \overline{\mathbb{T}}_\emptyset^{(2)}$ is an isomorphism of complete intersections. There exists an inteter r and a collection Q of r primes such that $R_Q^{(2)}$ may be generated by r elements as an $\mathcal{O}$-algebra. The following diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}[[S_1, \dots, S_r]] & \longrightarrow & \mathcal{O}[\Delta_Q] & & \\ \big\downarrow & & \big\downarrow & & \\ \mathcal{O}[[X_1, \dots, X_r]] & \longrightarrow & \overline{R}_\emptyset^{(2)} & \xrightarrow{\phi_\emptyset^{(2)}} & \overline{\mathbb{T}}_\emptyset^{(2)} \end{array}$$

where the horizontal maps are surjections, and the vertical maps are chosen so that the image of $\mathfrak{a}$ in $R_\emptyset^{(2)}$ is trivial. Now impose the extra condition

HM4: J_Q is free over $\mathcal{O}[\Delta_Q]$ and $J_Q / \mathfrak{a}\, J_Q \simeq J_\emptyset$.

Then $\overline{R}_\emptyset^{(2)} \simeq \overline{\mathbb{T}}_\emptyset^{(2)}$ and $J_\emptyset$ is free over $R_\emptyset^{(2)}$.

Step 2: Reduction to the Minimal Case. Denote $\mathrm{ad}^0\rho_{f,n} = \mathrm{ad}^0\rho_f \otimes_{\mathcal{O}} \lambda^{-n}\,\mathcal{O}/\mathcal{O}$ so that change in size from $\Phi_\emptyset$ to Φ_Σ satisfies

$$\frac{\#\Phi_\Sigma}{\#\,\Phi_\emptyset} \leq= \lim_{n\to\infty} \prod_{p\in\Sigma} \# \, H^0\left(G_p, \, \mathrm{ad}^0\rho_{f,n}(1)\right) = \prod_{p\in\Sigma} c_p.$$

Impose the extra condition

HM3: $\#\Omega_\Sigma / \#\,\Omega_\emptyset \geq \prod_{p\in\Sigma} c_p^2$ where $\Omega_\Sigma = J_\Sigma / \left(J_\Sigma[\mathfrak{p}_\Sigma] + J_\Sigma[I_\Sigma]\right)$.

Then $R_\Sigma^{(2)} \simeq \mathbb{T}_\Sigma^{(2)}$ as desired.

Step 3: Construction of Hecke Modules. We construct the modules using modular curves. ☐